المجلة الإلكترونية الشاملة متعددة التخصصات
العدد التاسع والثلاثون شهر (٩)٢٠٢١

Characterizations of Inner Product Spaces through some Geometrical .Theorems of Cyclic Quadrilateral and Trapezoid

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.Kingdom of Saudi Arabia, October – 2020

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:Abstract

In this paper, we discuss various generalizations of geometrical theorems of cyclic quadrilateral and trapezoid into real normed spaces. Current proofs are introduced, and some new characterizations of inner product spaces are obtained with some applications.

Keywords: Inner product spaces, Normed spaces**,** cyclic quadrilateral, trapezoid, isosceles trapezoid, norm, norm derivative, Euclidean geometry.

مستخلص:

في هذه الورقة، تمت مناقشة تعميمات مختلفة لبعض النظريات الهندسية ذات الصلة بالشكل الرباعي الدائري وشبه المنحرف داخل فضاء الضرب الداخلي للمتجهات، مما نتج عنها براهين حديثة، أسهمت في إيجاد تشخيصات جديدة لهذا الفضاء مع بعض التطبيقات.

مصطلحات البحث:
فضاء الضرب الداخلي، الفضاء المنظم، الرباعي الدائري، شبه المنحرف، شبه المنحرف المتساوي الساقين، النظيم، مشتقة النظيم، هندسة اقليدس.

1- Introduction:

In the Euclidean geometry, a quadrilateral inscribed in a circle and trapezoid theorems, found greatest interest by the researchers in classic spaces, so through them we will study some theorems to characterize the inner product spaces, based on the norm derivatives. First, we deal with Ptolemy's theorem, which states: "If a quadrilateral is inscribed in a circle then the product of the measures of its diagonals is equal to the sum of the products of the measures of the pairs of opposite sides".

(Consider A, B, C, and D in areal plane see Fig. (1

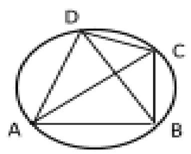

Figure SEQ Figure * ARABIC 1

$$\overline{AC} \bullet \overline{BD} = \overline{AB} \bullet \overline{DC} + \overline{AD} \bullet \overline{BC} \quad (1)$$

There are many known proofs to Ptolemy's Theorem based on classic spaces, in particular, using some trigonometric proof [1], [2] and [11], using similarities of triangle [11], using complex numbers [11], metric relation of circumcenter [3] and [4],using isomorphic triangle [5],[11],using trigonometric, circle geometry, and transformation geometry [6].

On the other side, we deal with a trapezoid theorem, which states: "the sum of squares of diagonals is equal to the sum of squares of non- parallel sides and the sum of twice the product of parallel sides".

We can write this theorem as follows**:**

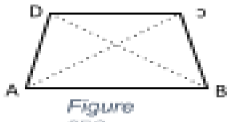

Figure SEQ Figure * ARABIC 2

$$\overline{AC}^2 + \overline{BD}^2 = \overline{AD}^2 + \overline{BC}^2 + 2\,\overline{AB} \bullet \overline{DC} \qquad (2)$$

The problem of finding necessary and sufficient geometrical conditions for a normed space to be inner product space it is still an open one. So, this theorem and others which associated with trapezoid sides measurements have been examined by many studies with different proofs see [10], [11], [12], [13].

Then by studying of these articles, that concern with cyclic quadrilateral, and trapezoid, and through the investigation in this area, the researcher revealed a gap in this field, which increased the researcher's motivations of high demand for a study and prompted him to conduct a research. The researcher adopted analytical descriptive approach through all theorems. Therefore, this paper aims to study various generalizations of geometrical theorems of cyclic quadrilateral and trapezoid into real normed spaces to investigate new characterizations of inner product spaces. Therefore, this paper tries to answer the following two questions:

i. What are the various generalizations of geometrical theorems in Euclidean plane of cyclic quadrilateral and trapezoid into real normed spaces?

ii. How new characterizations of inner product space can be investigated via generalizations of geometrical theorems of cyclic quadrilateral and trapezoid into real normed spaces?

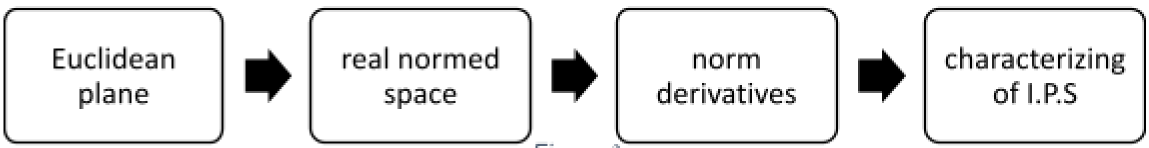

Figure 3

This study is unique one compared to the previous studies because it addresses some Euclidean geometrical theorems, and finds generalizations appropriate to them, into real normed spaces to establish some diagnoses of inner product space based on the norm derivative mapping.

2- Preliminaries

Definition 2.1: a real vector space X is called an inner product space if there is a real valued function $\langle \cdot, \cdot \rangle$ on $X \times X$ that satisfies the following four properties for all $x, y, z,$ in X and $\alpha \in R$:

i. $\langle x, x \rangle$ is nonnegative and $\langle x, x \rangle = 0$ if and only if $x = 0$.
ii. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$.
iii. $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$.
iv. $\langle x, y \rangle = \langle y, x \rangle$.

An inner product $\langle \cdot, \cdot \rangle$ defined on $X \times X$ induces the norm, so, all inner product spaces are normed linear spaces when the norm is induced by inner product, one says that the norm derives from an inner product.

Definition 2.2: In order to translate (formulate) these theorems into a real normed space, we will consider the two mappings $\rho'_{\pm}: X \times X \to R$ defined by:

$$\rho'_{\pm}(x, y) = \frac{\|x+ty\|^2 - \|x\|^2}{2t} \,. \, x. \, y \in X.$$

Proposition 2.3:

If $(X, \langle .,. \rangle)$ is a real inner product space then, both, ρ'_{+}, ρ'_{-} coincide with $\langle .,. \rangle$.

Proof:

$$\rho'_{\pm}(x, y) = \frac{\|x+ty\|^2 - \|x\|^2}{2t} \,.$$

$$= \lim_{t \to 0^{\pm}} \frac{\langle x+ty, x+ty \rangle - \langle x, x \rangle}{2t}.$$

$$= \lim_{t \to 0^{\pm}} \frac{\|x\|^2 + 2t\langle x, y \rangle + t^2\|y\|^2 - \|x\|^2}{2t}.$$

$$= \lim_{t \to 0^{\pm}} \frac{t\left(2\langle x, y \rangle + t\|y\|^2\right)}{2t}.$$

$$= \langle x, y \rangle.$$

The mappings $\rho'_{\pm}$ play a crucial role in this paper, so we give several propositions of these functions, which we used for different characterizations of inner product spaces. Indeed when the norm derives from an inner product space $(E, \langle .,. \rangle)$then

$\rho'_{\pm}(x, y) = \langle x, y\rangle$.We quote here some elementary properties concerning the functions $\rho'_{\pm}$ as follows: see [8+9].

1. $\rho'_{\pm}(x, y) = \|x\|^2$, and $\left|\rho'_{\pm}(x, y)\right| \le \|x\| \bullet \|y\|$;
2. $\rho'_{\pm}(\alpha x, y) = \rho'_{\pm}(x, \alpha y) = \alpha\rho'_{\pm}(x, y), x, y \in X, \alpha \in R, x \ge 0$;
3. $\rho'_{\pm}(x, \alpha x + y) = \alpha\|x\|^2 + \rho'_{\pm}(x, y), x, y \in X, \alpha \in R$;
4. $\rho'_{-}(x, y) \le \rho'_{+}(x, y), x, y \in X$;
5. $\rho'_{+}(x, y) = \rho'_{+}(y, x), \rho'_{-}(x, y) = \rho'_{-}(x, y), x, y \in X$;

If any of the following two conditions is verified, then the norm in X derives from an inner product space i.e. X is an inner product space

1. $\rho'_{+}(x, y) = \rho'_{+}(y, x)$, For all x, y in X.
2. $\rho'_{+}(u, v) = \rho'_{+}(v, u)$, For all u, v unit vectors in X.

3- Discussions:

In this section, we present some new characterizations of inner product spaces. Now let us consider a property of cyclic quadrilateral $ABCD$ in the real plane, E, F, G, be three points on the lengths AB, AC, AD, consequently, such that $AEFG$ be a parallelogram, then see fig (4),

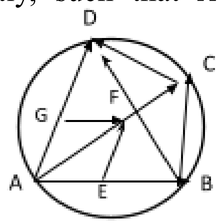

Figure 4

$$\overline{AF} \bullet \overline{AC} = \overline{AE} \bullet \overline{AB} + \overline{AG} \bullet \overline{AD} \tag{3}$$

In order to translate equation (3) into a real normed space, we consider $\vec{x}, \vec{y}, \vec{z},$ and $\vec{w}$ in X for all $\lambda, \beta, \alpha < 1$, then equation (3) becomes:

$$\lambda \|\vec{w}\|^2 = \beta\|\vec{x}\|^2 + \alpha\|\vec{y}\|^2 \tag{4}$$

Proposition 3.1: let $(X, \| \bullet \|)$ be a real normed space. Then X is an inner product space if, and only if for all vectors $\vec{x}, \vec{y}, \vec{z},$ and $\vec{w}$ in X, and for all $\lambda, \beta, \alpha < 1$, equation (4) holds.

Proof: $$\lambda \|\vec{w}\|^2 = \lambda \langle \vec{w}, \vec{w}\rangle$$

$$= \langle \lambda \vec{w}, \vec{w} \rangle$$

$$= \langle \beta \vec{x} + \alpha \vec{y}, \vec{w} \rangle$$

$$= \langle \beta \vec{x} + \alpha \vec{y}, \vec{x} + \vec{z} \rangle$$

$$= \beta \|\vec{x}\|^2 + \langle \beta \vec{x}, \vec{z} \rangle + \langle \alpha \vec{y}, \vec{x} \rangle + \langle \alpha \vec{y}, \vec{z} \rangle$$

$$= \beta \|\vec{x}\|^2 + \langle \lambda \vec{w}, \vec{z} \rangle + \langle \alpha \vec{y}, \vec{x} \rangle$$

$$= \beta \|\vec{x}\|^2 + \langle \lambda \vec{w}, \vec{z} \rangle + \langle \alpha \vec{y}, \vec{y} - \vec{BD} \rangle$$

$$= \beta \|\vec{x}\|^2 + \langle \lambda \vec{w}, \vec{z} \rangle + \alpha \|\vec{y}\|^2 - \langle \alpha \vec{y}, \vec{BD} \rangle$$

$$= \beta \|\vec{x}\|^2 + \alpha \|\vec{y}\|^2 + \langle \lambda \vec{w}, \vec{z} \rangle - \langle \alpha \vec{y}, \vec{BD} \rangle$$

But, $\triangle AFG \sim \triangle BDC$, then $\frac{\overline{AF}}{\overline{BD}} = \frac{\overline{AG}}{\overline{BC}}$

$$\|\lambda \vec{w}\| \bullet \|\vec{z}\| = \|\alpha \vec{y}\| \bullet \|\vec{BD}\|$$

$$\lambda \frac{\langle \vec{w}, \vec{z} \rangle}{\cos\cos \theta} = \alpha \frac{\langle \vec{y}, \vec{DB} \rangle}{\cos\cos \phi}, \text{ but } \quad \cos cos\, \theta = \cos cos\, \phi$$

$$\langle \lambda \vec{w}, \vec{z} \rangle = \langle \alpha \vec{y}, \vec{BD} \rangle$$

$$\langle \lambda \vec{w}, \vec{z} \rangle - \langle \alpha \vec{y}, \vec{BD} \rangle = 0.$$

Hence, the proposition is proved, and then X is an inner product space.

Ptolemy's Theorem in the Real Normed Space:

In order to translate equation (1) moving from the classic space into a real normed space, we consider the vectors: $\vec{x}, \vec{y}, \vec{z}$, and $\vec{w}$, such that:

$\vec{AF} = \lambda \vec{w}, \quad \lambda \in R^*$.

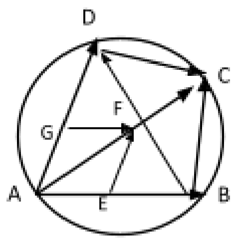

Figure 5

$\vec{AE} = \beta\vec{x}, \quad \beta \in R^*.$

$\vec{AG} = \alpha\vec{y}, \quad \alpha \in R^*$, see fig (5),

Then, equation (1) becomes as follows:

$$\|\vec{w}\| \bullet \|\vec{y} - \vec{x}\| = \|\vec{x}\| \bullet \|\vec{w} - \vec{y}\| + \|\vec{y}\| \bullet \|\vec{w} - \vec{x}\| \tag{5}$$

Theorem3.2: let $(X, \| \bullet \|)$ be a real normed space. Then X is an inner product space if, and only if for all vectors $\vec{x}, \vec{y}, \vec{z}, \vec{w}$ in X, and for all $\lambda, \beta, \alpha < 1$, equation (5) holds.

Proof. From proposition 3.1, we have

$$\lambda \|\vec{w}\|^2 = \beta\|\vec{x}\|^2 + \alpha\|\vec{y}\|^2$$

$$\|\vec{w}\| \bullet \|\lambda \vec{w}\| = \|\vec{x}\| \bullet \|\beta \vec{x}\| + \|\vec{y}\| \bullet \|\alpha \vec{y}\| \ \ldots\ldots\ldots\ldots \text{(I)}$$

Since, $\triangle AFG \sim \triangle BDC$, then

$$\frac{\overline{AF}}{\overline{BD}} = \frac{\overline{FG}}{\overline{DC}} = \frac{\overline{AG}}{\overline{BC}}.$$

$$\frac{\|\lambda \vec{w}\|}{\|\vec{y}-\vec{x}\|} = \frac{\|\beta \vec{x}\|}{\|\vec{w}-\vec{y}\|} = \frac{\|\alpha \vec{y}\|}{\|\vec{w}-\vec{x}\|}$$

$$\frac{\|\vec{y}-\vec{x}\|}{\|\lambda \vec{w}\|} = \frac{\|\vec{w}-\vec{y}\|}{\|\beta \vec{x}\|} = \frac{\|\vec{w}-\vec{x}\|}{\|\alpha \vec{y}\|} \ \ldots\ldots\ldots\ldots \text{(II)}$$

Multiply (I) by (II), we get:

$$\|\vec{w}\| \bullet \|\lambda \vec{w}\| \bullet \frac{\|\vec{y}-\vec{x}\|}{\|\lambda \vec{w}\|} = \|\vec{x}\| \bullet \|\beta \vec{x}\| \bullet \frac{\|\vec{w}-\vec{y}\|}{\|\beta \vec{x}\|} + \|\vec{y}\| \bullet \|\alpha \vec{y}\| \bullet \frac{\|\vec{w}-\vec{x}\|}{\|\alpha \vec{y}\|}.$$

Hence, the theorem is proved, and X is an inner product space.

On Sides and Diagonals of Trapezoid in an Inner Product Space:

In addition, let $ABCD$ be a trapezoid, considering vectors $\vec{x}, \vec{y}, \vec{w}$, for $\lambda > 0$ and $\vec{x} \neq \vec{y}$ as in fig (6), we have: $\vec{AC} = \vec{w}$,

$\vec{BD} = \vec{y} - \vec{x}, \ \alpha \in R^*.$

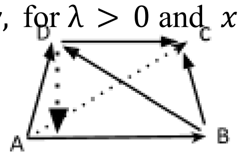

Figure 6

Then equation (2) into an inner product space becomes as follows:

$$\|\vec{w}\|^2 + \|\vec{y} - \vec{x}\|^2 = \|\vec{y}\|^2 + \|\vec{w} - \vec{x}\|^2 + 2\|\vec{x}\|\|\vec{w} - \vec{y}\| \tag{6}$$

Theorem3.3: let $(X, \|\bullet\|)$ be a real normed space. Then X is an inner product space if, and only if for all vectors $\vec{x}, \vec{y}, \vec{w}$, such that, $\alpha > 0$, equation (6) holds.

Proof. By using cosine formula, and from fig. (6) We get:

$$\|\vec{w}\|^2 = \|\vec{x}\|^2 + \|\vec{w} - \vec{x}\|^2 - 2\|\vec{x}\| \bullet \|\vec{w} - \vec{x}\| \cos \cos B \quad (7)$$

$$\|\vec{y} - \vec{x}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\| \bullet \|\vec{y}\| \cos \cos A \quad (8)$$

Then, by adding these two equations (7), (8), we have the following:

$$\|\vec{w}\|^2 + \|\vec{x} - \vec{y}\|^2$$

$$= \|\vec{y}\|^2 + \|\vec{w} - \vec{x}\|^2 + 2\|\vec{x}\|^2 - 2\|\vec{x}\| \bullet \|\vec{w} - \vec{x}\| \cos \cos B - 2\|\vec{x}\| \bullet \|\vec{y}\| \cos \cos A$$

$$= \|\vec{y}\|^2 + \|\vec{w} - \vec{x}\|^2 + 2\|\vec{x}\|\left(\|\vec{x}\| - \|\vec{w} - \vec{x}\| \cos \cos B - \|\vec{y}\| \cos \cos A\right). \quad (9)$$

However, from fig. (6), we get:

$$\|\vec{w} - \vec{y}\| = \|\vec{x}\| - \|\vec{w} - \vec{x}\| \cos \cos B - \|\vec{y}\| \cos \cos A \quad (10)$$

By substituting (10) in (9), hence the proof is followed, and X is an inner product space.

Isosceles trapezoid: Moreover, using the isosceles property for the trapezoid we have:

$$\vec{x} = \alpha\left(\vec{w} - \vec{y}\right), \quad \alpha > 0$$

$$\|\vec{Y} + \vec{Z}\|^2 + \|\vec{X} + \vec{Y}\|^2 = \|\vec{X}\|^2 + \|\vec{Z}\|^2 + 2\lambda\|\vec{Y}\|^2 . \text{ and}$$

$$\|\vec{y}\| = \|\vec{w} - \vec{x}\| .$$

Then by substituting in equation (6), we get:

$\|\frac{\vec{x}}{\alpha} + \vec{y}\|^2 + \|\vec{y} - \vec{x}\|^2 = 2\|\vec{y}\|^2 + \frac{2\|\vec{x}\|^2}{\alpha}$, Which implies to the following:

$$\alpha^2\left(\|\vec{x}\|^2 - 2\langle x, y\rangle\right) + 2\alpha\left(\langle x, y\rangle - \|\vec{x}\|^2\right) + \|\vec{x}\|^2 = 0,$$

Which is a quadratic equation, and by using the general formula, we get the following solutions: see [12]

Case (1): α =1, which means that the trapezoid can be inscribed in a circle and becomes a rectangle which attains parallelogram equality as follows:

$$\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2$$

Case (2): $\alpha = \frac{\|\vec{x}\|^2}{\|\vec{x}\|^2 - 2\langle\vec{x}, \vec{y}\rangle}$, now, we substitute the value of α in equation (6) as follows:

$$\|\frac{\vec{x}}{\lambda}+\vec{y}\|^2+\|\vec{y}-\vec{x}\|^2=2\|\vec{y}\|^2+2\|\vec{x}\|\frac{\|\vec{x}\|}{\lambda}$$

$$\|\frac{\vec{x}\left(\|\vec{x}\|^2-2\langle\vec{x},\vec{y}\rangle\right)}{\|\vec{x}\|^2}+\vec{y}\|^2+\|\vec{y}-\vec{x}\|^2=2\|\vec{y}\|^2+\frac{2\|\vec{x}\|^2\left(\|\vec{x}\|^2-2\langle\vec{x},\vec{y}\rangle\right)}{\|\vec{x}\|^2}$$

$$\|\vec{x}\left(\|\vec{x}\|^2-2\langle\vec{x},\vec{y}\rangle\right)+\|\vec{x}\|^2\bullet\vec{y}\|^2+\|\vec{x}\|^4\bullet\|\vec{y}-\vec{x}\|^2=$$

$$=2\|\vec{x}\|^4\bullet\|\vec{y}\|^2+2\|\vec{x}\|^6-4\|\vec{x}\|^4\langle\vec{x},\vec{y}\rangle.$$

$$\|\vec{x}\left(\|\vec{x}\|^2-2\rho'_+(x,y)\right)+\|\vec{x}\|^2\bullet\vec{y}\|^2+\|\vec{x}\|^4\bullet\|\vec{x}-\vec{y}\|^2=$$

$$=2\|\vec{x}\|^4\bullet\|\vec{y}\|^2+2\|\vec{x}\|^6-4\|\vec{x}\|^4\rho'_+(x,y)\qquad(11)$$

Theorem3.4: Let $(X,\|\bullet\|)$ be a real normed space. Then X is an inner product space if, and only if for all vectors $\vec{x},\vec{y},\vec{w}$ in X, equation (11) holds.

Proof. Substitute x by tx with $t>0$ in equation (11), we get:

$$\|\|\vec{x}\|^2\bullet\vec{y}-2\vec{x}\rho'_+(x,y)\|^2+\|\vec{x}\|^4\bullet\|\vec{y}\|^2=2\|\vec{x}\|^4\bullet\|\vec{y}\|^2.$$

Then by using $\vec{u}$ instead of $\vec{x}$ and $\vec{v}$ instead of $\vec{y}$, where $\vec{u}$and $\vec{v}$ are unit vectors, then this equality becomes: $\|\vec{v}-2\vec{u}\rho'_+(u,v)\|^2=1,$

Then we have: $4\rho'_+(u,v)^2-4\rho'_+(u,v)\cdot\rho'_+(v,u)=0$

$$\rho'_+(u,v)\left(\rho'_+(u,v)-\rho'_+(v,u)\right)=0,$$

Then we conclude that,

$$\rho'_+(u,v)=0,\text{ or }\rho'_+(u,v)=\rho'_+(v,u)$$

Since this equality hold for $\vec{u},\vec{v}\in S_x$, then X is an inner product space by condition (2).

On Trapezoid Based on a Triangle in an Inner Product Space:

Now again we will translate equation (2) into an inner product space considering vectors $\vec{x}, \vec{y}, \alpha\vec{x}, \alpha\vec{y}$, for $\alpha > 0$ and $\vec{x} \neq \vec{y}$ as in fig (7), so we have:

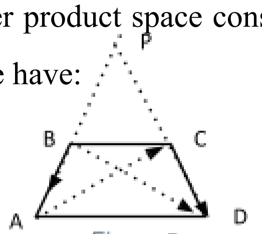

Figure 7

$\vec{AC} = \vec{y} - \alpha\vec{x}, \quad \alpha > 1$

$\vec{BD} = \alpha\vec{y} - \vec{x}, \quad \alpha > 1.$

Then we translate equation (2) as follows: see [11]

$$||\vec{y} - \alpha\vec{x}||^2 + ||\alpha\vec{y} - \vec{x}||^2 = ||\alpha\vec{x} - \vec{x}||^2 + ||\alpha\vec{y} - \vec{y}||^2 + 2\alpha||\vec{y} - \vec{x}||^2.$$

$$||\vec{y} - \alpha\vec{x}||^2 + ||\alpha\vec{y} - \vec{x}||^2 = (\alpha - 1)^2\left(||\vec{x}||^2 + ||\vec{y}||^2\right) + 2\alpha||\vec{y} - \vec{x}||^2. \qquad (12)$$

Theorem 3.5: let $(X, ||\bullet||)$ be a real normed space. Then X is an inner product space if, and only if for all vectors $\vec{x}, \vec{y}, \alpha\vec{x}, \alpha\vec{y}$, and for all, $\alpha > 1$, equation (12) holds.

Proof.

$$||\vec{y} - \alpha\vec{x}||^2 = \alpha^2||\vec{x}||^2 + ||\vec{y}||^2 - 2\alpha\langle\vec{y}, \vec{x}\rangle$$

$$||\alpha\vec{y} - \vec{x}||^2 = \alpha^2||\vec{y}||^2 + ||\vec{x}||^2 - 2\alpha\langle\vec{y}, \vec{x}\rangle$$

$$||\vec{y} - \alpha\vec{x}||^2 + ||\alpha\vec{y} - \vec{x}||^2 = \alpha^2||\vec{x}||^2 + ||\vec{y}||^2 - 2\alpha\langle\vec{y}, \vec{x}\rangle + \alpha^2||\vec{y}||^2 + ||\vec{x}||^2 - 2\alpha\langle\vec{y}, \vec{x}\rangle - 2\alpha||\vec{y} - \vec{x}||^2 + 2\alpha||\vec{y}$$

$$= \alpha^2||\vec{x}||^2 + ||\vec{x}||^2 + \alpha^2||\vec{y}||^2 + ||\vec{y}||^2 - 4\alpha\langle\vec{y}, \vec{x}\rangle - 2\alpha||\vec{y}||^2 - 2\alpha||\vec{x}||^2 + 4\alpha\langle\vec{y}, \vec{x}\rangle + 2\alpha||\vec{y} - \vec{x}||^2$$

$$=$$

$$\alpha^2||\vec{x}||^2 - 2\alpha||\vec{x}||^2 + ||\vec{x}||^2 + \alpha^2||\vec{y}||^2 - 2\alpha||\vec{y}||^2 + ||\vec{y}||^2 + 2\alpha||\vec{y} - \vec{x}||^2.$$

$$= \left(\alpha^2 - 2\alpha + 1\right)||\vec{x}||^2 + \left(\alpha^2 - 2\alpha + 1\right)||\vec{y}||^2 + 2\alpha||\vec{y} - \vec{x}||^2.$$

$$= \left(\alpha^2 - 2\alpha + 1\right)\left(||\vec{x}||^2 + ||\vec{y}||^2\right) + 2\alpha||\vec{y} - \vec{x}||^2.$$

Reciprocally, assuming the hypothesis for unitary vectors u and v instead of x and y, i.e.: u, v in S_E, we have that for all u, v in S_E and $\alpha > 0$:

$$\frac{\|\vec{v}-\alpha\vec{u}\|^2-\|\vec{u}\|^2}{2\alpha}+\frac{\|\alpha\vec{v}-\vec{u}\|^2-\|\vec{u}\|^2}{2\alpha}$$

$$=\frac{(\alpha-1)^2\left(\|\vec{u}\|^2+\|\vec{v}\|^2\right)}{2\alpha}+\frac{2\alpha\|\vec{v}-\vec{u}\|^2}{2\alpha}-\frac{\left(\|\vec{u}\|^2+\|\vec{v}\|^2\right)}{2\alpha}$$

$$\rho'_+(v,u)+\rho'_+(u,v)=\frac{\left(\|\vec{u}\|^2+\|\vec{v}\|^2\right)(\alpha^2-2\alpha)}{2\alpha}+\|\vec{v}-\vec{u}\|^2$$

$$=\lim_{\alpha\to 0^+}\frac{\left(\|\vec{u}\|^2+\|\vec{v}\|^2\right)(\alpha-2)}{2}+\|\vec{v}-\vec{u}\|^2$$

$$2\rho'_+(u,v)=\|\vec{v}-\vec{u}\|^2-\left(\|\vec{u}\|^2+\|\vec{v}\|^2\right)$$

$$\rho'_+(u,v)=\frac{\|\vec{v}-\vec{u}\|^2-2}{2}$$

The symmetry of ρ'_+ for unitary vectors hold, and this is sufficient condition to characterize X as an inner product space.

The Height's Measurement of Trapezoid in an Inner Product Space:

If X is an inner product space (I.P.S) and $\vec{x}$ and $\vec{y}$ are two independent vectors in $X-\{0\}$, in the trapezoid of vertics A, B, C and D, the length of the height (h) over $\overline{AB}$ is given by:

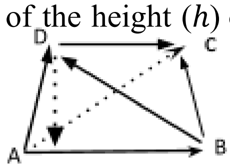

Figure 8

$$h=\frac{2}{|\overline{AB}-\overline{DC}|}\sqrt{(S-\overline{DC})(S-\overline{AB})(S-\overline{DC}-\overline{BC})(S-\overline{DC}-\overline{AD})} \quad (13)$$

Where S is the semi perimeter, $S=\frac{\overline{AB}+\overline{BC}+\overline{DC}+\overline{AD}}{2}$, then equation (13) can be translated as follows:

$$h=\frac{2}{\left|\|\vec{y}-\vec{x}\|-\|\vec{z}\|\right|}\sqrt{\left(S-\|\vec{z}\|\right)(S-\|\vec{x}\|)\left(S-\|\vec{z}\|-\|\vec{w}\|\right)\left(S-\|\vec{z}\|-\|\vec{y}\|\right)} \quad (14)$$

Several new characterizations of inner product space have been obtained when dealing with special properties of the function$h(x,y)$, see [13], [14].

In the case of isosceles trapezoid, we have:

$$\|\vec{y}\| = \|\vec{w}\|,$$

$$\|\vec{z}\| = \|\vec{y}-\vec{x}\| - 2\|\frac{\|\vec{y}\|^2 - \rho'_+(y,x)\bullet(\vec{y}-\vec{x})}{\|\vec{y}-\vec{x}\|^2}\|,$$

$$S = \|\vec{y}-\vec{x}\| + \|\vec{y}\| - \|\frac{\|\vec{y}\|^2 - \rho'_+(y,x)\bullet(\vec{y}-\vec{x})}{\|\vec{y}-\vec{x}\|^2}\|,$$

So, equation (14) translated as follows:

$$h(x,y) = \sqrt{\|\vec{y}\|^2 - \|\frac{\|\vec{y}\|^2 - \rho'_+(y,x)\bullet(\vec{y}-\vec{x})}{\|\vec{y}-\vec{x}\|^2}\|^2} \tag{15}$$

Theorem 3.6: : Let $(X, \|\bullet\|)$ be a real normed space with dim $X \geq 2$,then X is an inner product space if, and only if, equation (15) holds, for all independent vectors $\vec{x}, \vec{y}$, in X.

Proof:

Cosider the function $h(x,y) = \vec{y} + \frac{\|\vec{y}\|^2 - \rho'_+(x,y)\bullet(\vec{x}-\vec{y})}{\|\vec{x}-\vec{y}\|^2}$, see [14], defined for all $\vec{x}$, $\vec{y} \in X \cdot X$, $\vec{x} \neq \vec{y}$, $\|h(x,y)\|$ gives the usual height over $\overline{AB}$. Substitute $y = tz$, $t > 0$, in the definition of $h(x,y)$, then divide both sides by t and take $t \to 0$, as follows:

$$\frac{\|h(x,tz)\|}{t} = \|tz + \frac{\|tz\|^2 - \rho'_+(x,tz)\bullet(\vec{x}-tz)}{\|\vec{x}-tz\|^2}\| / t$$

$$= \|\left(z + \frac{t\|z\|^2 - \rho'_+(x,z)\bullet(\vec{x}-tz)}{\|\vec{x}-tz\|^2}\right)\|$$

$$\frac{\|h(x,tz)\|}{t} = \|z - \frac{\rho'_+(x,z)\bullet(\vec{x})}{\|\vec{x}\|^2}\| \tag{16}$$

On the other hand by (15):

$$\frac{\|h(x,tz)\|}{t} = \sqrt{\|tz\|^2 - \|\frac{\|tz\|^2 - \rho'_-(x,tz)\bullet(\vec{x}-tz)}{\|\vec{x}-tz\|^2}\|^2} / t$$

$$= \sqrt{\|z\|^2 - \|\frac{t\|z\|^2 - \rho'_-(x,z)\bullet(\vec{x}-tz)}{\|\vec{x}-tz\|^2 \bullet}\|^2}$$

$$\frac{\|h(x,tz)\|}{t} = \sqrt{\|z\|^2 - \frac{\rho'_-(x,z)^2}{\|\vec{x}\|^2}}, \tag{17}$$

Then from (16), and (17) ,we get the following equality:

$$\|z - \frac{\rho'_+(x,z)\bullet(\vec{x})}{\|\vec{x}\|^2}\| = \sqrt{\|z\|^2 - \frac{\rho'_-(x,z)^2}{\|\vec{x}\|^2}}$$

By squaring both sides we get:

$$\|z - \frac{\rho'_+(x,z)\bullet(\vec{x})}{\|\vec{x}\|^2}\|^2 = \|z\|^2 - \frac{\rho'_-(x,z)^2}{\|\vec{x}\|^2},$$

$$\|\frac{z}{\|z\|} - \rho'_+\left(\frac{x}{\|x\|}, \frac{z}{\|z\|}\right) \bullet \frac{x}{\|x\|}\|^2 = \|\frac{z}{\|z\|}\|^2 - \rho'_-\left(\frac{x}{\|x\|}, \frac{z}{\|z\|}\right)^2, \tag{18}$$

Let $u = \frac{x}{\|x\|}$, and $v = \frac{z}{\|z\|}$, such that u, v are two unit vectors in X, then equation(18) be as follows:

$$\|v - \rho'_+(u,v)\cdot u\|^2 = 1 - \rho'_-(u,v)^2, \tag{19}$$

Substitute v by $\frac{u+v}{\|u+v\|}$, we obtain:

$$\|\frac{u+v}{\|u+v\|} - \rho'_+\left(u, \frac{u+v}{\|u+v\|}\right)\cdot u\|^2 = 1 - \rho'_-\left(u, \frac{u+v}{\|u+v\|}\right)^2,$$

$$\|u + v - \rho'_+(u, u + v)\cdot u\|^2 = \|u + v\|^2 - \rho'_-(u, u + v)^2$$

$$\|v + u(1 - \rho'_+(u, u + v)\|^2 = \|u + v\|^2 - \rho'_-(u, u + v)^2$$

$$\|v + u(1 - 1 - \rho'_+(u, v))\|^2 = \|u + v\|^2 - (1 + \rho'_-(u, v))^2$$

$$\|v - \rho'_+(u, v)u\|^2 = \|u + v\|^2 - 1 - \rho'_-(u, v)^2 - 2\rho'_-(u, v)$$

$$1 + \rho'_+(u, v)^2 - 2\rho'_+(u, v)^2 = \|u + v\|^2 - 1 - \rho'_-(u, v)^2 - 2\rho'_-(u, v)$$

$$1 - \rho'_+(u, v)^2 = \|u + v\|^2 - 1 - \rho'_-(u, v)^2 - 2\rho'_-(u, v)$$

$$1 = \|u + v\|^2 - 1 - 2\rho'_-(u, v)$$

$$2 + 2\rho'_-(u, v) = \|u + v\|^2$$

$$2 + 2\rho'_{+}(u, v) = \|v + u\|^2$$

$$2 + 2\rho'_{-}(u, v) = 2 + 2\rho'_{-}(v, u)$$

$$\rho'_{-}(u, v) = \rho'_{-}(v, u).$$

4 – Applications of Ptolemy's theorem: Here, we have some applications of Ptolemy's theorem on real normed spaces through some mathematical identities.

From equation (5), and when a cyclic quadrilateral being a trapezoid, this leads to the following corollary:

Corollary 4.1: Let $(X, \| \bullet \|)$ be a real normed space with dim, $X \geq 2$ then X is an inner product space if, and only if, for all independent vectors $\vec{x}, \vec{y}, \vec{w}$ in X such that the following equation holds:

$$\|\vec{x}\left(\|\vec{x}\|^2 - 2\rho'_{+}(x, y)\right) + + \vec{y} \bullet \|\vec{x}\|^2 \| \bullet \|\vec{y} - \vec{x}\| =$$

$$= \|\vec{x}\|^2\left(\|\vec{x}\|^2 - 2\rho'_{+}(x, y)\right) + \vec{y}\,\|\vec{x}\left(\|\vec{x}\|^2 - 2\rho'_{+}(x, y)\right) + \|\vec{x}\|^2\left(\vec{y} - \vec{x}\right)\| \qquad (20)$$

First application:

Suppose that a circle contain point A of parallelogram $APQR$ and intersects side $\overline{AP}$, $\overline{AR}$ and diagonals $\overline{AQ}$, $\overline{DB}$ in points B , D and C respectively,

See fig. (9). so, we have the following property: see [7]

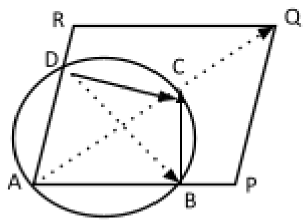

Figure 9

$$\overline{AC} \bullet \overline{AQ} = \overline{AB} \bullet \overline{AP} + \overline{AD} \bullet \overline{AR} \qquad (20)$$

We consider the vectors $\vec{x}, \vec{y}, \vec{w}, \beta\,\vec{x}, \alpha\vec{y}, \lambda\vec{w}$, for all $\lambda, \beta, \alpha \in R^*$, then equation (20) becomes:

$$\|\vec{w}\| \bullet \|\lambda\vec{w}\| = \|\vec{x}\| \bullet \|\beta\,\vec{x}\| + \|\vec{y}\| \bullet \|\alpha\vec{y}\|$$

Which can be written as:

$$\lambda\,\|\vec{w}\|^2 = \beta\|\vec{x}\|^2 + \alpha\|\vec{y}\|^2 \qquad (21)$$

Corollary 4.2: let $(X, \|\bullet\|)$ be a real normed space. Then $\|\bullet\|$ is derived from an inner product space if, and only if for all vectors $\vec{x}, \vec{y}, \vec{z}, \vec{w}$ in X, and for all, $\lambda\ \beta,\ \alpha > 1$, equation (21) holds.

Proof: From Ptolemy's theorem, we have:

$$\|\vec{w}\| \bullet \|\vec{y} - \vec{x}\| = \|\vec{x}\| \bullet \|\vec{y} - \vec{w}\| + \|\vec{y}\| \bullet \|\vec{w} - \vec{x}\|$$

Since, $\triangle BCD \sim \triangle AQP$, then $\frac{\overline{DB}}{\overline{AQ}} = \frac{\overline{DC}}{\overline{AP}} = \frac{\overline{BC}}{\overline{PQ}}$.

$$\frac{\|\vec{y}-\vec{x}\|}{\|\lambda\,\vec{w}\|} = \frac{\|\vec{y}-\vec{w}\|}{\|\beta\,\vec{x}\|} = \frac{\|\vec{w}-\vec{x}\|}{\|\alpha\,\vec{y}\|}$$

From the two steps, we have:

$$\|\vec{w}\| \bullet \|\lambda\,\vec{w}\| \bullet \frac{\|\vec{y}-\vec{x}\|}{\|\lambda\,\vec{w}\|} = \|\vec{x}\| \bullet \|\beta\,\vec{x}\| \bullet \frac{\|\vec{y}-\vec{w}\|}{\|\beta\,\vec{x}\|} + \|\vec{y}\| \bullet \|\alpha\,\vec{y}\| \bullet \frac{\|\vec{w}-\vec{x}\|}{\|\alpha\,\vec{y}\|}$$

$$\lambda\,\|\vec{w}\|^2 = \beta\|\vec{x}\|^2 + \alpha\|\vec{y}\|^2$$

Hence, the corollary is proved.

Second Application:

a square $ABCD$ is inscribed a circle and P is a point on the arc $\widehat{BC}$ of the circle, then we have the following property:

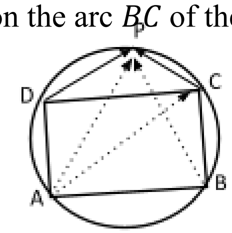

Figure 10

$$\frac{\overline{PA}+\overline{PC}}{\overline{PB}+\overline{PD}} = \frac{\overline{PD}}{\overline{PA}}. \qquad (22)$$

It can be translated into a real normed space by considering $\vec{x}$, $\vec{y}$ as vectors as follows:

$$\frac{\|2\vec{x}+\vec{y}\|+\|\vec{y}\|}{\|\vec{x}+\vec{y}\|+\|\vec{x}+\vec{y}\|} = \frac{\|\vec{x}+\vec{y}\|}{\|2\vec{x}+\vec{y}\|}.$$

$$\frac{\|2\vec{x}+\vec{y}\|+\|\vec{y}\|}{2\|\vec{x}+\vec{y}\|} = \frac{\|\vec{x}+\vec{y}\|}{\|2\vec{x}+\vec{y}\|}. \qquad (23)$$

Corollary 4.3: let $(X, \|\bullet\|)$ be a real normed space. Then $\|\bullet\|$ is derived from an inner product space if, and only if for all vectors $\vec{x}, \vec{y}$ in X, equation (23) holds.

Proof: from the cyclic quadrilateral $ABCP$ and by using Ptolemy's theorem we get

$$\|2\vec{x}\| \bullet \|\vec{x} + \vec{y}\| = \|\vec{x}\| \bullet \|\vec{y}\| + \|\vec{x}\| \bullet \| 2\vec{x} + \vec{y}\|$$

Again, from $ABPD$ we have:

$$\|2\vec{x}\| \bullet \|2\vec{x} + \vec{y}\| = \|\vec{x}\| \bullet \|\vec{x} + \vec{y}\| + \|\vec{x}\| \bullet \| \vec{x} + \vec{y}\|$$

Then by divide the first equation by the second one, we get

$$\frac{\|2\vec{x}\|\|\vec{x}+\vec{y}\|}{\|2\vec{x}\|+\|2\vec{x}+\vec{y}\|} = \frac{\|\vec{x}\|\bullet\|\vec{y}\|+\|\vec{x}\|\bullet\| 2\vec{x}+\vec{y}\|}{\|\vec{x}\|\bullet\| \vec{x}+\vec{y}\|+\|\vec{x}\|\bullet\| \vec{x}+\vec{y}\|}$$

Which implies that:

$$\frac{\|\vec{x}+\vec{y}\|}{\|2\vec{x}+\vec{y}\|} = \frac{\|2\vec{x}+\vec{y}\|+\|\vec{y}\|}{2\|\vec{x}+\vec{y}\|}.$$

Third Application (Golden Ratio):

Let $ABCDE$ be a regular pentagon in a circle, then the golden ratio r is given by:

$$r = \frac{d}{a} = \frac{diagonal}{side} = \frac{1+\sqrt{15}}{2}.$$

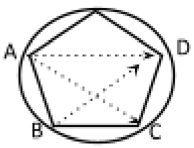

Figure 11

Which can be written in the normed space as:

$$\frac{\|\vec{x}+\vec{y}\|}{\|\vec{x}\|} = \frac{1+\sqrt{15}}{2}. \qquad (24)$$

Corollary 4.4: let $(X, \|\bullet\|)$ be a real normed space. Then $\|\bullet\|$ is derived from an inner product space if, and only if for all vectors $\vec{x}, \vec{y}$ in X, equation (24) holds.

Proof: by using Ptolemy's theorem in the shape$ABCD$, we have

$$\| \vec{w} + \vec{y}\| \bullet \|\vec{x} + \vec{y}\| = \|\vec{x}\| \bullet \|\vec{w}\| + \|\vec{y}\| \bullet \| \vec{z} + \vec{r}\|$$

From regularity, we have:

$$\|\vec{x} + \vec{y}\| = \| \vec{z} + \vec{r}\| = \| \vec{w} + \vec{y}\|$$

$$\|\vec{x}\| = \|\vec{w}\| = \|\vec{y}\|$$

Then we have: $$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{x}\| \bullet \| \vec{x} + \vec{y}\|$$

$$\left(\frac{\|\vec{x}+\vec{y}\|}{\|\vec{x}\|}\right)^2 - \frac{\|\vec{x}+\vec{y}\|}{\|\vec{x}\|} - 1 = 0.$$

By solving this quadratic equation, we have the following solution,

$$\frac{\|\vec{x}+\vec{y}\|}{\|\vec{x}\|} = \frac{1+\sqrt{15}}{2}.$$

Which is golden ratio.

Fourth Application.

Ptolemy's theorem also provides some trigonometric identities; here we prove the addition and the subtraction formula for sine and cosine rule.

Addition Formula: Let $ABCD$ be a cyclic quadrilateral such that the side AC serve as a diameter and $\overline{AC} = 1$, then from the fig. (12) We have:

$$\|\vec{w}\| = 1.$$

$$\|\vec{w} - \vec{y}\| = \sin sin\, \alpha.$$

$$\|\vec{y}\| = \cos cos\, \alpha.$$

$$\vec{w} - \vec{x} = \sin sin\, \beta\ .$$

$$\|\vec{x}\| = \cos cos\, \beta.$$

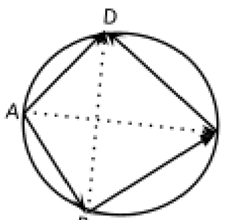

Figure 12

In addition, from $\triangle ADB$, we have: $\frac{\|\vec{x}-\vec{y}\|}{\sin sin\,(\alpha+\beta)} = \frac{\|\vec{y}\|}{\sin sin\,(90-\alpha)}$.

Which implies that: $\sin sin\,(\alpha + \beta) = \|\vec{x} - \vec{y}\|$

Then by substituting in the following Ptolemy's theorem:

$$\|\vec{w}\| \bullet \|\vec{x} - \vec{y}\| = \|\vec{x}\| \bullet \|\vec{w} - \vec{y}\| + \|\vec{y}\| \bullet \|\ \vec{w} - \vec{x}\|$$

We get: $\sin sin\,(\alpha + \beta) = \sin sin\, \alpha\cdot \cos cos\, \beta + \cos cos\, \alpha\cdot \sin sin\, \beta$

Which called the addition formula of sine.

Subtraction Formula: Let $ABCD$ be a cyclic quadrilateral such that the side BC serve as a diameter and $\overline{BC} = 1$, then from the fig. (13) We have:

$$\|\vec{w} - \vec{x}\| = 1.$$

$$\|\vec{w}\| = \sin sin\, \alpha.$$

$$\|\vec{X}\| = \cos cos\, \alpha.$$

$$\|\vec{w} - \vec{y}\| = \sin sin\, \beta\ .$$

$$\|\vec{x} - \vec{y}\| = \cos cos\, \beta.$$

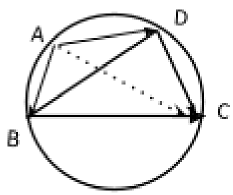

Figure 13

And from $\triangle ADB$, we have:

$$\frac{\|\vec{y}\|}{\sin sin\,(\alpha-\beta)} = \frac{\|\vec{y}\|}{\sin sin\,(90-\beta)}.$$

Which implies that: $\sin sin\,(\alpha - \beta) = ||\vec{y}||$

Then by substituting in the following Ptolemy's theorem:

$$||\vec{w}|| \bullet ||\vec{x} - \vec{y}|| = ||\vec{x}|| \bullet ||\vec{w} - \vec{y}|| + ||\vec{y}|| \bullet ||\vec{w} - \vec{x}||$$

We get:

$\sin sin\,(\alpha - \beta) = \sin sin\,\alpha \cdot \cos cos\,\beta - \cos cos\,\alpha \cdot \sin sin\,\beta$.

Which called the subtraction formula of sine.

Cosine rule: let ABC be any arbitrary triangle. Then we have the following relation:

$$a^2 = b^2 + c^2 - 2bccosA \qquad (25)$$

Proof: Let $ABCD$ be an isosceles trapezoid such that, then see fig.(14)

$||\vec{y}|| = ||\vec{w} - \vec{x}||$

$\overline{AC}|| = \vec{w}|| = ||\vec{x} - \vec{y}||$

$||\vec{y} - \vec{w}|| = ||\vec{x}|| - 2||\vec{w} - \vec{x}|| \cos cos\left(\theta_1 + \theta_2\right)$

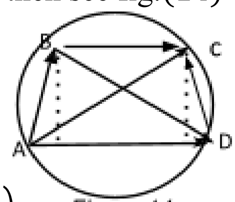

Figure 14

Then from Ptolemy's theorem, we have:

$$||\vec{w}|| \bullet ||\vec{x} - \vec{y}|| = ||\vec{x}|| \bullet ||\vec{w} - \vec{y}|| + ||\vec{y}|| \bullet ||\vec{w} - \vec{x}||$$

$$= ||\vec{x}||\left(||\vec{x}|| - 2||\vec{w} - \vec{x}|| \cos cos\left(\theta_1 + \theta_2\right)\right) + ||\vec{y}|| \bullet ||\vec{w} - \vec{x}||$$

$$||\vec{w}||^2 = ||\vec{x}||^2 + ||\vec{w} - \vec{x}||^2 - 2||\vec{w}||||\vec{w} - \vec{x}|| \cos cos\left(\theta_1 + \theta_2\right). \qquad (26)$$

Which we called formula of cosine in the real normed space.

Corollary 4.5: let ABC be Congruent triangle such that A, B. C, and Q on a circle, then

$$\overline{AQ} = \overline{QB} + \overline{QC}, \qquad (27)$$

Which translated as: $||\vec{w}|| = ||\vec{w} - \vec{x}|| + ||\vec{w} - \vec{y}||$ (28)

Proof: Then from Ptolemy's theorem, we have:

$$||\vec{w}|| \bullet ||\vec{x} - \vec{y}|| = ||\vec{x}|| \bullet ||\vec{w} - \vec{y}|| + ||\vec{y}|| \bullet ||\vec{w} - \vec{x}||$$

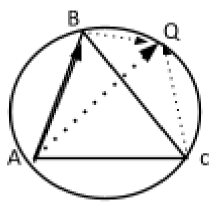

Figure 15

But: $\|\vec{y}\| = \|x - \vec{y}\| = \|\vec{x}\|$

$$\frac{\|\vec{w}\|\bullet\|x-\vec{y}\|}{\|x-\vec{y}\|} = \frac{\|\vec{x}\|\bullet\|\vec{w}-\vec{y}\|}{\|\vec{x}\|} = \frac{\|\vec{y}\|\bullet\|\vec{w}-\vec{x}\|}{\|\vec{y}\|}$$

$$\|\vec{w}\| = \|\vec{w} - \vec{x}\| + \|\vec{w} - \vec{y}\|$$

Hence, the corollary proved.

Acknowledgement:

The author would like to express his Gratitude's to the ministry of education and the deanship of scientific research – Najran University – Kingdom of Saudi Arabia for their financial and Technical support under code number (NU/ESCI/17/048).

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