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Legendre Polynomials as Terms of Three-parameter Mittag-Leffler Function

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Abstract

In this paper, the relationship between Legendre polynomials and some special cases of the 3-parameter Mittag-Leffler function is discussed. The generalization of the relationship between 3-parameter Mittag-Leffler function and Legendre polynomials $P_n(x)$ also is discussed by using some properties of Legendre polynomials.

Keywords

Three-parameter Mittag-Leffler function, Legendre Polynomials, Fractional Calculus

1- Introduction

Legendre Polynomials were introduced by Adrien Marie Legendre in 1782 [1], these polynomials have many properties; the most important property of Legendre polynomials is orthogonality of these polynomials. Legendre polynomials have many physical applications and Legendre differential equation is a very important ordinary differential equation in engineering and physics [2].

2- The problem and objective:

Many researchers have studied Legendre polynomials and their applications in many fields. This study aims to determine the relationship between Legendre polynomials and fractional calculus and some special cases of Mittag-Leffler functions.

3- The hypothesis:

This paper hypothesizes that there are strong relationships between 3-parameter Mittag-Leffler function and Legendre polynomials.

Legendre polynomials:

Legendre polynomial $P_n(x)$ is written as

$$P_n(x) = \frac{1}{2^n}\sum_{r=0}^{k} \frac{(-1)^r\,(2n-2r)!\,x^{n-2r}}{r!(n-2r)!(n-r)!} \qquad (1)$$

$k = [\frac{n}{2}\,, \quad if \;\; n \;\; is\; an\; even\; number \quad \frac{n-1}{2}\,, \quad if \;\; n \;\; is\; an\; odd\; number$

We have another representation of Legendre polynomial $P_n(x)$ called Rodrigues formula, it is written as

$$P_n(x) = \frac{1}{2^n\,n!}\,\frac{d^n}{dx^n}\left[x^2 - 1\right]^n \qquad (3)$$

Recursive formulas of Legendre polynomials are

$$1)\; (n+1)\,P_{n+1}(x) - (2n+1)\,x\,P_n(x) + n\,P_{n-1}(x) = 0 \qquad (4)$$

$$2)\; P'_{n+1}(x) - x\,P'_n(x) = (n+1)\,P_n(x) \qquad (5)$$

We can prove previous formulas from equation (1).

We have a strong relation between Legendre polynomials and fractional calculus, especially the 3-parameter Mittag-Leffler function.

This paper is structured as follows: Section 2 represents the 3-parameter Mittag-Leffler function and some special cases and discusses the relationship between these cases and some Legendre polynomials. In section 3 we will show how to generalize representation of Legendre polynomials as functions of some special cases of 3-parameter Mittag-Leffler function.

Some Special cases of 3-parameter Mittag-Leffler Function:

Gustaf Mittag-Leffler introduced the Mittag-Leffler functions in the beginning of twentieth century [3-18]. The basic Mittag-Leffler function is written as

$$E_{\alpha}(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(1+\alpha j)} \tag{6}$$

if $\alpha = 1$, we get the expansion of exponential function

$$E_1(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(1+j)} = \sum_{j=0}^{\infty} \frac{y^j}{j!} = e^y \tag{7}$$

We have another types of Mittag-Leffler functions, and second type in 2-parameter Mittag-Leffler function is written as

$$E_{\alpha,\beta}(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(\beta+\alpha j)} \tag{8}$$

if $\beta = 1$, we get the basic Mittag-Leffler function.

Third type of Mittag-Leffler functions is 3-parameter Mittag-Leffler function, it is written as

$$E^{\gamma}_{\alpha,\beta}(y) = \sum_{j=0}^{\infty} \frac{\Gamma(\gamma+j)\, y^j}{j!\,\Gamma(\gamma)\Gamma(\beta+\alpha j)} \tag{9}$$

if $\gamma = 1$, we get 2-parameter Mittag-Leffler function

$$E^{1}_{\alpha,\beta}(y) = \sum_{j=0}^{\infty} \frac{\Gamma(1+j)\, y^j}{j!\,\Gamma(1)\Gamma(\beta+\alpha j)} = \sum_{j=0}^{\infty} \frac{j!\, y^j}{j!\Gamma(\beta+\alpha j)}$$

$$= \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(\beta+\alpha j)} = E_{\alpha,\beta}(y) \tag{10}$$

We can find more special cases of 3-parameter Mittag-Leffler function, we can let $\alpha =- 2$, $\beta = 3$ and $\gamma = \frac{-3}{2}$ in equation (9), we get

$$E^{\frac{-3}{2}}_{-2,3}(y) = \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{-3}{2}+j\right)(y)^j}{j!\,\Gamma\left(\frac{-3}{2}\right)\Gamma(3-2j)} \tag{11}$$

if $y = x^2$, we get

$$E^{\frac{-3}{2}}_{-2,3}\left(x^2\right) = \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{-3}{2}+j\right)\left(x^2\right)^j}{j!\,\Gamma\left(\frac{-3}{2}\right)\Gamma(3-2j)}$$

$$= \frac{\Gamma\left(\frac{-3}{2}+0\right)\left(x^2\right)^0}{0!\,\Gamma\left(\frac{-3}{2}\right)\Gamma(3-2(0))} + \frac{\Gamma\left(\frac{-3}{2}+1\right)\left(x^2\right)^1}{1!\,\Gamma\left(\frac{-3}{2}\right)\Gamma(3-2(1))} + 0 + 0 + \ldots$$

$$= \frac{1}{\Gamma(3)} + \frac{\frac{-3}{2}\Gamma\left(\frac{-3}{2}\right) x^2}{\Gamma\left(\frac{-3}{2}\right)\Gamma(1)} = \frac{1}{2!} - \frac{3}{2}\frac{x^2}{0!}$$

$$= \frac{1}{2} - \frac{3}{2}x^2 \qquad (12)$$

We can substitute $n = 2$ in equation(1), we get

$$P_2(x) = \frac{1}{2^2}\sum_{r=0}^{\frac{2}{2}} \frac{(-1)^r\,(2(2)-2r)!\,x^{2-2r}}{r!(2-2r)!(2-r)!}$$

$$= \frac{(-1)^0\,(4-2(0))!\,x^{2-2(0)}}{4\;\;0!(2-2(0))!(2-0)!} + \frac{(-1)^1\,(4-2(1))!\,x^{2-2(1)}}{4\;\;1!(2-2(1))!(2-1)!}$$

$$= \frac{4!\,x^2}{4\;\;2!\;\;2!} - \frac{2!}{4\;\;0!\;\;1!} = \frac{4\times3\times2\times1\;\;x^2}{4\times2\times1\times\;2\times1} - \frac{2\times1}{4}$$

$$= \frac{3}{2}x^2 - \frac{1}{2} \qquad (13)$$

Also we can substitute $n = 2$ in equation(3), we get the same result in equation (13)

$$P_2(x) = \frac{1}{2^2\,2!}\frac{d^2}{dx^2}\left[x^2 - 1\right]^2 = \frac{1}{8}\frac{d}{dx}\left[4\,x\left(x^2 - 1\right)\right]$$

$$= \frac{1}{8}\left(12x^2 - 4\right) = \frac{3}{2}x^2 - \frac{1}{2} \tag{14}$$

From (12) and (13) or (12) and (14) , we get

$$P_2(x) = -E^{\frac{-3}{2}}_{-2,3}\left(x^2\right) \tag{15}$$

Similarly, we can find the relationship between $P_1(x)$ and $E^{\frac{-1}{2}}_{-2,1}(x)$, by putting

$\alpha = -2$, $\beta = 1$ and $\gamma = \frac{-1}{2}$ in equation(9), we get

$$E^{\frac{-1}{2}}_{-2,1}(x) = \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{-1}{2}+j\right)(x)^j}{j!\,\Gamma\left(\frac{-1}{2}\right)\Gamma(1-2j)}$$

$$= \frac{\Gamma\left(\frac{-1}{2}+0\right)(x)^0}{0!\,\Gamma\left(\frac{-1}{2}\right)\Gamma(1-2(0))} + 0 + 0 + \ldots$$

$$= \frac{1}{\Gamma(1)} = \frac{1}{0!} = 1 \tag{16}$$

Now, by substituting $n = 1$ in equation(1), we get

$$P_1(x) = \frac{1}{2^1}\sum_{r=0}^{\frac{1-1}{2}} \frac{(-1)^r\,(2(1)-2r)!\,x^{1-2r}}{r!(1-2r)!(1-r)!}$$

$$= \frac{(-1)^0\,(2-2(0))!\,x^{1-2(0)}}{2\;\;0!(1-2(0))!(1-0)!} = \frac{2!}{2}x = x \tag{17}$$

Also we can substitute $n = 1$ in equation (3), we get the same result in equation (17)

$$P_1(x) = \frac{1}{2^1\,1!}\,\frac{d}{dx}\left[x^2 - 1\right]^1 = \frac{1}{2}\times 2x = x \qquad (18)$$

From (16) and (17) or (16) and (18), we get

$$P_1(x) = x\,E^{\frac{-1}{2}}_{-2,1}(x) \qquad (19)$$

By putting $n = 0$ in equation(1), we get

$$P_0(x) = \frac{1}{2^0}\sum_{r=0}^{\frac{0}{2}} \frac{(-1)^r\,(2(0)-2r)!\,x^{0-2r}}{r!(0-2r)!(0-r)!}$$

$$= \frac{(-1)^0\,(2(0)-2(0))!\,x^{0-2(0)}}{0!(0-2(0))!(0-0)!} = 1 \qquad (20)$$

Also, we can substitute $n = 0$ in equation(3), we get

$$P_0(x) = \frac{1}{2^0\,1!}\left[x^2 - 1\right]^0 = 1 \qquad (21)$$

From (16) and (20) or (16) and (21), we get

$$P_0(x) = E^{\frac{-1}{2}}_{-2,1}(x) \qquad (22)$$

Representation of Legendre Polynomials as terms of Some Special Cases of 3-parameter Mittag-Leffler Function.

In previous section we got the relationship between $P_0(x)$, $P_1(x)$, $P_2(x)$ and some special cases of 3-parameter Mittag-Leffler function. Now, we have very important question "What is the relationship between 3-parameter Mittag-Leffler function and $P_n(x)$, $\forall n \in N$ in general? ", to answer this question we will discuss the relationship between $P_n(x)$, $P_{n+1}(x)$ and $P_{n-1}(x)$ from equation (4) , by putting $= 1$, we get

$$(1 + 1)\, P_{1+1}(x) - (2(1) + 1)\, x\, P_1(x) + (1)\, P_{1-1}(x) =$$

$$2\, P_2(x) - 3\, x\, P_1(x) + P_0(x) = 0 \qquad (23)$$

From (15) , (19) and (22) , the equation (23) becomes

$$- 2E^{\frac{-3}{2}}_{-2,3}\left(x^2\right) - 3\, x \left(x\, E^{\frac{-1}{2}}_{-2,1}(x)\right) + E^{\frac{-1}{2}}_{-2,1}(x) = 0 \qquad (24)$$

then

$$E^{\frac{-3}{2}}_{-2,3}\left(x^2\right) = \frac{1}{2}\left(1 - 3\, x^2\right) E^{\frac{-1}{2}}_{-2,1}(x) \qquad (25)$$

By putting $n = 2\, in$ (4) , we get

$$(2 + 1)\, P_{2+1}(x) - (2(2) + 1)\, x\, P_2(x) + (2)\, P_{2-1}(x) =$$

$$3\, P_3(x) - 5\, x\, P_2(x) + 2P_1(x) = 0 \qquad (26)$$

then

$$P_3(x) = -\frac{1}{3}\left(2P_1(x) - 5\, x\, P_2(x)\right) \qquad (27)$$

From (15) , (19) , (22) and (25) , the equation (27) becomes

$$P_3(x) = -\frac{1}{3}\left(2xE^{\frac{-1}{2}}{}_{-2,1}(x) + 5\, x\left(\frac{1}{2}\left(1 - 3\, x^2\right)E^{\frac{-1}{2}}{}_{-2,1}(x)\right)\right)$$

$$= -\frac{1}{3}\left(2x + \frac{5\, x}{2} - \frac{15}{2}\, x^3\right)E^{\frac{-1}{2}}{}_{-2,1}(x)$$

$$= \left(\frac{5}{2}\, x^3 - \frac{3\, x}{2}\right)E^{\frac{-1}{2}}{}_{-2,1}(x) \qquad (28)$$

4- Results and Discussion:

This section generalizes the relationship between 3-parameter Mittag-Leffler function and Legendre polynomials, we can represent

$$P_m(x) = \frac{1}{m}\left((2m - 1)\, x\, P_{m-1}(x) + (1 - m)\, P_{m-2}(x)\right)E^{\frac{-1}{2}}{}_{-2,1}(x) \qquad (29)$$

where $P_{m-1}(x)$ and $P_{m-2}(x)$ are functions of x and some special cases of 3-parameter Mittag-Leffler function, we can apply all properties of Legendre polynomials in these special cases of 3-parameter Mittag-Leffler function

5- Conclusions and future limitations:

The basic objective of this paper was to find the relationship between Legendre polynomials $P_n(x)$ for all natural numbers and some special cases of 3-parameter Mittag-Leffler function by using some properties of Legendre polynomials.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Conflicts of interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Bell, W. W., Special functions for Scientists and Engineers, D. Van Nostrand Company LTD, London, 1967.

[2] Bender, C. M., Orszag, S. A., Advanced Mathematical Methods for Scientists and Engineers,1st-Springer, (1999).

[3] Capelas de Oliveira, E., Mainardi, F., Vaz Jr, J., Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics, The European Physical Journal, Special Topics 193(2011), 161-171.

[4] Djrbashian, M.M., Bagian, A., On integral representations and measures associated with Mittag-Leffler type functions, Izv. Akdd. Nauk Armjanskvy SSR, Mathematica 10(1975), 483-508.

[5] Gorenflo, R., Kilbas, A. A., Mainardi, F., Rogosin, S. V., Mittag-Leffler Functions, Related Topics and Applications, Springer, New York, 2014.

[6] Gorenflo, R., Luchko, Yu., Mainardi, F., Analytical properties and applications of the Wright function, 2 No 4(1999) 383-414.

[7] Haubold, H. J., Mathai, A. M., An Introduction to Fractional Calculus, Mathematics Research Development, New York, 2017.

[8] Haubold, H. J., Mathai, A. M., Saxena, R. K., Mittag-Leffler functions and their applications, Journal of Applied Mathematics, ID298628(2011), 51 pages.

[9] Hilfer, R., Seybold, H.J., Computation of the generalized Mittag-Leffler function and its inverse in the complex plane, Integral Transforms and Special Functions, 17No. 9(2006), 637-652.

[10] Kilbas, A.A., Saigo, M., On Mittag-Leffler type functions, fractional calculus operators and solution of integral equations, Integral Transforms and Special Functions, 4(1996), 355-370.

[11] Mathai, A. M., A pathway to matrix-variate gamma and normal densities, Linear Algebra and its Application, 396(2005), 317-328.

[12] Mathai, A. M., Haubold, H. J., A pathway from Bayesian statistical analysis to super statistics, Applied Mathematics and Computations, 218(2011), 799-804.

[13] Mathai, A.M., Some properties of Mittag-Leffler functions and matrix-variate analogues, Fractional Calculus and Applied Analysis, 13(2)(2010), 113-132.

[14] Mittag-Leffler, G., Mittag-Leffler, Sur la représentation analytique d'une fonction monogène(cinquième note), Acta Mathematica, vol. 29,no. 1, pp. 101-181, 1905.

[15] Mittag-Leffler, G. M., Sur la nouvelle fonction $E_{\alpha}(x)$Comptes de L'Académie des Sciences, vol. 137, pp. 554-558, 1903.

[16] Mittag-Leffler, G. M., Une généralisation de L'intégrale de Laplace-Abel, L'Académie des Sciences Serie II, vol. 137, pp. 537-539, 1903.

[17] Nair, S. S., Statistical Distributions Connected with Pathway Model and Generalized Fractional Integral Operator, Ph.D Thesis, Banaras Hindu University, 2012.

[18] Pillai, R.N., On Mittag-Leffler functions and related distributions, Ann. Inst. Statist. Math., 42(1990) 157-161.