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Partial Malliavin Calculus

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Abstract:

This paper aimed at investigating the partial Malliavin calculus, The operators $D_{\mathcal{H}}, \delta_{\mathcal{H}}$ and $L_{\mathcal{H}}$ associated with the projection on $\mathcal{H}$, The existence of A conditional density.

Keywords: *Malliavin calculus, Gaussian process, Covariance function, Orthogonal complement, Operators $D_{\mathcal{H}}, \delta_{\mathcal{H}}$ and $L_{\mathcal{H}}$*

I. INTRODUCTION

Let H be a real separable Hilbert space. Suppose that $W = \{ w(h), h \in H \}$ is a Gaussian process with zero mean and covariance function given by $E\big(w(h)w(\mathrm{g})\big) = \langle h, \mathrm{g}\rangle$, defined in some probability space $(\Omega, \mathcal{F}, p)$. This means that DF can be considered as an element of $L^2(\Omega; H \otimes E)$.

We introduce the operator δ, defined on H-valued smooth functionals $G = \mathrm{g}\big(w(h_1), \dots, w(h_n)\big)h$ as follows

$$\delta(G) = \mathrm{g}\big(w(h_1), \dots, w(h_n)\big)w(h)$$

$$-\sum_{j=1}^{n}(\partial_j \mathrm{g})\big(w(h_1), \dots, w(h_n)\big)\cdot\langle h_j, h\rangle. \qquad (1.1)$$

Notice that $\delta(\mathrm{G})$ is a real valued random variable.

We recall the following basic properties of these operators:

(i) The chain rule: If $F = (F_1, \dots, F_m) \in \mathbb{D}_{2,1}(\mathbb{R}^m)$ and $\phi : \mathbb{R}^m \to \mathbb{R}$ is a C^1 function with bounded partial derivatives then

$$D\phi(F) = \sum_{i=1}^{m}(\partial_i\phi)(F)DF_i. \qquad (1.2)$$

(ii) The integration by parts formula: If G and F are smooth random variables taking values in H and $\mathbb{R}$, respectively, then:

$$E(\langle G, DF\rangle) = E(\delta(G)F). \qquad (1.3)$$

This means that δ is the dual of D. If we denote by $Dom\ \delta \subset L^2(\Omega; \mathrm{H})$ the domain of the operator δ considered as the dual of the unbounded operator D on $L^2(\Omega)$ (with domain $\mathbb{D}_{2,1}$), then formula (2.19)[56] holds for any $F \in \mathbb{D}_{2,1}$ and $G \in Dom\ \delta$.

(iii) $LF = \delta DF$, for any F in the domain of L as an operator on $L^2(\Omega)$.

Definition (1): We define the partial derivative operator $D_{\mathcal{H}} : \mathbb{D}_{2,1} \to L^2(\Omega; \mathrm{H})$ as the projection of D on $\mathcal{H}$, namely, for any $F \in \mathbb{D}_{2,1}$,

$$D_{\mathcal{H}}F = \prod\nolimits_{\mathcal{H}}(DF) = \prod\nolimits_{K(\omega)}(DF)\,(\omega).$$

Some properties of this derivative:

(i) Let $F = f\big(w(h_1), \dots, w(h_k)\big)$ be a smooth functional. Then

$$DF = \sum_{i=1}^{k}(\partial_i f)\big(w(h_1), \dots, w(h_k)\big)h_i\text{ , and}$$

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$$D_{\mathcal{H}}F=\sum_{i=1}^{k}(\partial_i f)\big(w(h_1),\dots,w(h_k)\big)\prod\nolimits_K h_i.$$

Note that for any $h\in H$ we have

$$\langle D_{\mathcal{H}}F,h\rangle=\langle DF,\prod\nolimits_K h\rangle=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}F\left(\omega+\varepsilon\prod\nolimits_{K(\omega)}(h)\right).$$

(ii) The chain rule. Let $F_1,\dots,F_m\in\mathbb{D}_{2,1}$ and let $\phi:\mathbb{R}^m\to\mathbb{R}$ be a continuously differentiable function with bounded first derivatives. Then $\phi(F)\in\mathbb{D}_{2,1}$, and

$$D_{\mathcal{H}}\phi(F)=\sum_{i=1}^{m}(\partial_i\phi)(F)D_{\mathcal{H}}F_i.$$

In fact, it suffices to project on $K(\omega)$ the ordinary chain rule for the derivative operator.

It is well known that D is a closed operator on $\mathbb{D}_{2,1}$.

Definition (2): Set

$$DomL_{\mathcal{H}}=\{F\in\mathbb{D}_{2,1}:D_{\mathcal{H}}F\in Dom\ \delta\}=\{F\in\mathbb{D}_{2,1}:DF\in Dom\ \delta_{\mathcal{H}}\},$$

and for any $F\in DomL_{\mathcal{H}}$ we define

$$L_{\mathcal{H}}F=\delta_{\mathcal{H}}D_{\mathcal{H}}F=\delta D_{\mathcal{H}}F.$$

Properties of the operator $L_{\mathcal{H}}$:

(i) It follows from property (ii) of $\delta_{\mathcal{H}}$ that

$$L_{\mathcal{H}}\psi(F_1,\dots,F_m)=\delta\prod\nolimits_{\mathcal{H}}\left(\sum_{i=1}^{m}(\partial_i\psi)(F)DF_i\right)$$

$$=L\left(\sum_{i=1}^{m}(\partial_i\psi)(F)D_{\mathcal{H}}F_i\right)$$

$$=\sum_{i=1}^{m}(\partial_i\psi)(F)L_{\mathcal{H}}F_i-\sum_{i,j=1}^{m}\big(\partial_i\partial_j\psi\big)(F)\,\langle D_{\mathcal{H}}F_i,D_{\mathcal{H}}F_j\rangle,$$

Provided that the components of $F=(F_1,\dots,F_m)$ belong to $DomL_{\mathcal{H}}$, ψ is a smooth bounded function with bounded first and second partial derivatives, and $E(\|DF_i\|^4)<\infty, i=1,\dots,m$.

(ii) Under the condition of Lemma (2.2.2) [56], smooth functionals of the form $f\big(w(h_1),\dots,w(h_k)\big)$ belong to $DomL_{\mathcal{H}}$, and therefore, $DomL_{\mathcal{H}}$ is dense in $L^2(\Omega)$.

2. PRELIMINARIES

We derive two results regarding the existence of conditional densities. These results hold under relatively weak assumptions on the Malliavin derivatives but are restricted in other directions. For the first result the conditioning σ-field is restricted to be finitely smoothly generated. For the second result the last restriction is dropped, however it is assumed that the random variable for which the conditional density is obtained is one-dimensional (and not a finite dimensional vector). Both results are motivated by the work of Bouleau-Hirsch [66]. We consider stronger assumption on the partial Malliavin matrix. Also, conditions for the smoothness of the density will be considered.

We assume that H is a real separable Hilbert space and $W = \{w(h), h \in H\}$ is a Gaussian process.

Theorem (1): Let $G_1, \dots, G_n$ be elements of $\mathbb{D}_{2,1}$ satisfying $det\,\langle DG_i, DG_j\rangle > 0\ a.s.$ Set $\mathcal{H} = \{K(\omega)\,, \omega \in \Omega\}$ with $K(\omega) = \langle DG_i(\omega), i = 1, \dots, n\rangle^{\perp}$. Let $F = (F_1, \dots, F_m), F_i \in \mathbb{D}_{2,1}$and assume that

$$det\,\langle D_{\mathcal{H}}F_i, D_{\mathcal{H}}F_j\rangle > 0 \qquad a.s.$$

Then, there exists a conditional density for the law of F given the σ-field $\sigma\{G_1, \dots, G_n\}$.

Proof: Consider the augmented vector

$$(G_1, \dots, G_n, F_1, \dots, F_m).$$

Note that in order to prove the theorem it suffices to show that the augmented vector possesses a joint density. The determinant of the Malliavin matrix of the augmented vector is given by:

$$Q = det\begin{bmatrix} \langle DG_i, DG_j\rangle & \langle DG_i, DF_j\rangle \\ \langle DG_i, DF_j\rangle^T & \langle DF_i, DF_j\rangle \end{bmatrix}. \tag{2.1}$$

The result of Bouleau and Hirsch is that if the above determinant is a.s. non zero then the augmented vector has a probability density.

On the other hand, it was shown by lkeda, Shigekawa and Taniguchi (equation 3.29 of [57]) that

$$Q = det\big[\langle DG_i, DG_j\rangle\big] \cdot det\big[\langle D_{\mathcal{H}}F_i, D_{\mathcal{H}} \neq F_j\rangle\big] \tag{2.2}$$

where Q is as defined by (2.30)[56]. By our assumption this expression is positive and this completes the proof.

Theorem (2): Let $F \in \mathbb{D}_{2,1}$be a real valued random variable, and $\underline{G} = (G_i\,, i \geq 1), G_i \in \mathbb{D}_{2,1}$. Assume that $D_{\mathcal{H}}$ is a closed operator where $\mathcal{H}$ is induced by $\underline{G}$, that means, $\mathcal{H} = \{K(\omega)\,, \omega \in \Omega\}$ and $K(\omega) = \langle DG_i(\omega), i \geq 1\rangle^{\perp}$ (cf. Lemma (2.2.2)) [56]. If $\langle D_{\mathcal{H}}F, D_{\mathcal{H}}F\rangle > 0\ a.s.$, then F has a conditional density with respect to the sub- σ-field generated by $\underline{G}$.

Proof: Without any loss of generality we may assume that F is bounded, namely $|F| < 1$. Denote by $P_{\underline{G}}$ the probability law induced by $\underline{G}$ on $\mathbb{R}^{\infty}$. Then it suffices to show that the probability law induced by the vector $(F, \underline{G}\,)$ on $(-1,1) \times \mathbb{R}^{\infty}$, denoted by $P_{(F,\underline{G}\,)}$, is absolutely continuous with respect to the product measure $d\,\alpha dP_{\underline{G}}(\underline{x})$. In that case the Radon-Nikodym derivative

$$f(\alpha, \underline{x}\,) = \frac{dP_{(F,\underline{G}\,)}(\alpha, \underline{x})}{d\,\alpha dP_{\underline{G}}(\underline{x})} \tag{2.3}$$

will provide a version for the conditional density of F given $\underline{G} = \underline{x}$.

We have, therefore, to show that for any measurable function g: $(-1,1) \times \mathbb{R}^\infty \to [0,1]$ such that $\int g(\alpha, \underline{x}) d\, \alpha dP_{\underline{G}}(\underline{x}) = 0$ we have $E[g(F, \underline{G}\,)] = 0$. If g is such a function we have

$$\int g(\alpha, \underline{x}) d\, \alpha = 0 \qquad (2.4)$$

for almost all $\underline{x}$ with respect to the low of $\underline{G}$. Consequently, there exists a sequence of continuously differentiable functions with bounded derivatives $g^n: (-1,1) \times \mathbb{R}^n \to [0,1]$ such that $g^n(\alpha, x_1, \dots, x_n)$ converges to $g(\alpha, \underline{x})$ for almost all $(\alpha, \underline{x})$ with respect to the measure $dP_{(F,\underline{G}\,)}(\alpha, \underline{x}) + d\, \alpha dP_{\underline{G}}(\underline{x})$. Take

$$\psi^n(y, x_1, \dots, x_n) = \int_{-1}^{y} g^n(\alpha, x_1, \dots, x_n) d\alpha$$

and

$$\psi(y, \underline{x}) = \int_{-1}^{y} g(\alpha, \underline{x}) d\alpha \,.$$

Then $\psi^n(F, G_1, \dots, G_n) \in \mathbb{D}_{2,1}$ and

$$D[\psi^n(F, G_1, \dots, G_n)] = g^n(F, G_1, \dots, G_n) DF + \sum_{i=1}^{n} \frac{\partial \psi^n}{\partial x_i}(F, G_1, \dots, G_n) DG_i . \qquad (2.5)$$

We have

$$\psi^n(f, G_1, \dots, G_n) \to \psi(F, \underline{G}\,)$$

$a.s., as\ \ n \to \infty$, and in $L^2(\Omega)$ by dominated convergence. Because of (2.4) with $g(\alpha, \underline{x})$ nonnegative, it holds that $\psi(F, \underline{G}\,) = 0\ a.s.$ Now from (8)

$$D_{\mathcal{H}}\,[\psi^n(F, G_1, \dots, G_n)] = g^n(F, G_1, \dots, G_n) D_{\mathcal{H}} F, \qquad (2.6)$$

which converges a.s. to $g(F, \underline{G}\,) D_{\mathcal{H}} F$. Thus $g(F, \underline{G}\,) D_{\mathcal{H}} F = 0$ because $D_{\mathcal{H}}$ was assumed to be a closed operator, and, therefore, $g(F, \underline{G}\,) = 0\ \ a.s.$, because $\langle D_{\mathcal{H}} F, D_{\mathcal{H}} F \rangle\ > 0\ a.s.$, which completes the proof of the theorem.

Proposition (1): Suppose that $\{F_i\,, i \geq 1\}$ and $\{G_i\,, i \geq 1\}$ generate the same σ-field $\mathcal{G}$, and $F_i, G_i \in \mathbb{D}_{2,1}$ for any $i \geq 1$. Assume that the families $\mathcal{H}_F = \{\langle DF_i\,,\ i \geq 1\rangle^{\perp}\}$ and $\mathcal{H}_G = \{\langle DG_i\,,\ i \geq 1\rangle^{\perp}\}$ are such that $D_{\mathcal{H}_F}$ and $D_{\mathcal{H}_G}$ are closed operators. Then $\mathcal{H}_F = \mathcal{H}_G$.

Proof: It suffices to show that $DF \in \langle DG_i\,,\ i \geq 1\rangle$ for any $\mathcal{G}$-measurable $F \in \mathbb{D}_{2,1}$. There exists a sequence $\psi_n(G_1, \dots, G_n) \to F$ as $n \to \infty$, in $L^2(\Omega)$ and a.s. We may assume that the functions ψ_n are in $C_b^\infty(\mathbb{R}^n)$. Clearly $D_{\mathcal{H}_G}[\psi_n(G_1, \dots, G_n)] = 0$, since the projection is on the orthogonal to $\langle DG_i\,,\ i \geq 1\rangle$. So $D_{\mathcal{H}_G} F = 0$ a.s., because $D_{\mathcal{H}_G}$ is closed, and this implies that $DF \in \mathcal{H}_G^{\perp} = \langle DG_i\,,\ i \geq 1\rangle$.

Throughout this section we assume that $\mathcal{G} = \sigma\{G_i\,, i \geq 1\}$ is countably smoothly generated and $\mathcal{H} = \mathcal{H}_G$.

Proposition (2):

(a) Conditional integration by parts formula: For any $F \in \mathbb{D}_{2,1}$ and $u \in Dom\ \delta_{\mathcal{H}}$, we have

$$E(\langle u, D_{\mathcal{H}}F\rangle|\mathcal{G}) = E(F\delta_{\mathcal{H}}u|\mathcal{G})$$

(b) $L_{\mathcal{H}}$ is "conditionally self-adjoint": For any F, Q in the domain of $L_{\mathcal{H}}$,

$$E(QL_{\mathcal{H}}F|\mathcal{G}) = E(FL_{\mathcal{H}}Q|\mathcal{G}).$$

Proof: Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be a C^1-function bounded and with bounded derivatives. Set $R = \psi(G_1, \dots, G_m)$. Then by (2.21) [56]

$$E(FR\delta_{\mathcal{H}}u) = E(\langle D_{\mathcal{H}}(FR), u\rangle)$$

$$= E(\langle FD_{\mathcal{H}}R, u\rangle + \langle RD_{\mathcal{H}}F, u\rangle)$$

$$= E(R\langle D_{\mathcal{H}}F, u\rangle),$$

which proves the first part. The second part follows since

$$E(QL_{\mathcal{H}}F|\mathcal{G}) = E(Q\delta_{\mathcal{H}}D_{\mathcal{H}}F|\mathcal{G})$$

$$= E(\langle D_{\mathcal{H}}Q, D_{\mathcal{H}}F\rangle|\mathcal{G}) = E(FL_{\mathcal{H}}G|\mathcal{G}).$$

3. CLAIMS

Definition (3): Set $Dom\ \delta_{\mathcal{H}} = \{u \in L^2(\Omega; \mathrm{H}) : \prod_{\mathcal{H}} u \in Dom\ \delta\}$, and for any $u \in Dom\ \delta_{\mathcal{H}}$, set $\delta_{\mathcal{H}}u = \delta \prod_{\mathcal{H}} u$.

With this definition we have the following integration by parts formula:

$$E(F\delta_{\mathcal{H}}u) = E\left(F\delta \prod\nolimits_{\mathcal{H}} u\right)$$

$$= E\left(\langle DF, \prod\nolimits_{\mathcal{H}} u\rangle\right)$$

$$= (\langle D_{\mathcal{H}}F, u\rangle), \tag{3.1}$$

for any $u \in Dom\ \delta_{\mathcal{H}}$ and $F \in \mathbb{D}_{2,1}$.

Notice that the condition in Lemma (2.2.2) [56] implies that the H-valued smooth random variables belong to $Dom\ \delta_{\mathcal{H}}$. So, $Dom\ \delta_{\mathcal{H}}$ is a dense subset of $L^2(\Omega; \mathrm{H})$.

Some properties of the operator $\delta_{\mathcal{H}}$:

(i) Let $u \in Dom\ \delta_{\mathcal{H}}$, then it is clear from the definition that $\prod_{\mathcal{H}} u \in Dom\ \delta_{\mathcal{H}}$, and $\delta_{\mathcal{H}} \prod_{\mathcal{H}} u = \delta \prod_{\mathcal{H}} u = \delta_{\mathcal{H}}u$.

(ii) Let $u \in Dom\ \delta_{\mathcal{H}}$, and $F \in \mathbb{D}_{2,1}$. Then $Fu \in Dom\ \delta_{\mathcal{H}}$ and

$$\delta_{\mathcal{H}}(Fu) = F\delta_{\mathcal{H}}u - \langle u, D_{\mathcal{H}}F\rangle, \tag{3.2}$$

Provided that the right hand side is square integrable.

The proof is a direct consequence of the same result without $\mathcal{H}$(see [65]).

Lemma (1): If $\prod_{\mathcal{H}} h \in Dom\ \delta$ for all $h \in H$, then $D_{\mathcal{H}}$ is a closed operator on $\mathbb{D}_{2,1}$.

Proof: For any $F \in \mathbb{D}_{2,1}$, we can write using integration by parts

$$E(\langle h, D_{\mathcal{H}}F\rangle) = E\left(\langle\prod\nolimits_{\mathcal{H}} h\,, DF\rangle\right) = E\left(\delta\left(\prod\nolimits_{\mathcal{H}} h\right)F\right).$$

More generally, for any smooth H-valued random variable $G : \Omega \to \mathrm{H}$ like

$G = \sum_{i=1}^{m} \xi_i(\omega)h_i$, we have $\prod_{\mathcal{H}} G \in Dom\ \delta$ (since $\prod_{\mathcal{H}} h_i$ were assumed to be in $Dom\ \delta$, and the ξ_i are smooth), and

$$E(\langle G, D_{\mathcal{H}}F\rangle) = E\left(\langle\prod\nolimits_{\mathcal{H}} G\,, DF\rangle\right) = E\left(\delta\left(\prod\nolimits_{\mathcal{H}} G\right)F\right). \tag{3.3}$$

This implies that $D_{\mathcal{H}}$ is closed since

$$\left.\begin{matrix} F_n \xrightarrow{L^2(\Omega)} 0\,, & F_n \in \mathbb{D}_{2,1} \\ D_{\mathcal{H}}F_n \xrightarrow{L^2(\Omega;\mathrm{H})} \eta & \end{matrix}\right\} \Rightarrow \eta = 0.$$

In fact, setting $F = F_n$ in (3.3) and letting $n \to \infty$ yields the result.

Theorem (3): Let $F = (F_1, \dots, F_m)$ be a k-dimensional random vector verifying the following condition:

(i) $F_i \in \mathbb{D}_{2,1}$, $D_{\mathcal{H}}F_i \in Dom\ \delta$ and $\langle D_{\mathcal{H}}F_i, D_{\mathcal{H}}F_j\rangle \in \mathbb{D}_{2,1}$ for any $i, j = 1, \dots, k$.

(ii) The partial Malliavin matrix $\gamma_{\mathcal{H}}^{ij} = \langle D_{\mathcal{H}}F_i, D_{\mathcal{H}}F_j\rangle$ is invertible a.s.

Then there exists a conditional density for the law of F given the σ-algebra $\mathcal{G}$.

Proof: For any integer $N \geq 1$ we consider a function $\psi_N \in C_0^\infty(\mathbb{R}^m \otimes \mathbb{R}^m)$ ($C^\infty$and with compact support) such that

(a) $\psi_N(\sigma) = 1$ if $\sigma \in K_N$,

(b) $\psi_N(\sigma) = 0$ if $\sigma \notin K_{N+1}$, where

$K_N = \left\{\sigma \in \mathbb{R}^m \otimes \mathbb{R}^m : |\sigma^{ij}| \leq N\ \ for\ any\ i, j\ and\ |det\sigma| \geq \frac{1}{N}\right\}, i.e. K_N$ is a compact subset o

f $GL(m) \subset \mathbb{R}^m \otimes \mathbb{R}^m$.

We fix a function $\phi \in C_b^{\infty}(\mathbb{R}^m)$. Using the differentiation rules of the partial Malliavin calculus we deduce $\phi(F) \in \mathbb{D}_{2,1}$ and

$$D_{\mathcal{H}}\phi(F) = \sum_{i=1}^{m} (\partial_i\phi)(F) D_{\mathcal{H}}F_i.$$

Hence,

$$\langle D_{\mathcal{H}}\phi(F), D_{\mathcal{H}}F_j\rangle = \sum_{i=1}^{m} (\partial_i\phi)(F)\gamma_{\mathcal{H}}^{ij},$$

where $\gamma_{\mathcal{H}}^{ij}$ is as defined above in the statement of the theorem. Then, we have

$$E[\psi_N(\gamma_{\mathcal{H}})(\partial_i\phi)(F)|\mathcal{G}] = \sum_{j=1}^{m} E\left[\psi_N(\gamma_{\mathcal{H}})\langle D_{\mathcal{H}}\phi(F), D_{\mathcal{H}}F_j\rangle(\gamma_{\mathcal{H}}^{-1})^{ij}|\mathcal{G}\right]$$

$$= \sum_{j=1}^{m} E\left[\langle D_{\mathcal{H}}\left(\phi(F)(\gamma_{\mathcal{H}}^{-1})^{ij}\psi_N(\gamma_{\mathcal{H}})\right), D_{\mathcal{H}}F_j\rangle - \phi(F)\,\langle D_{\mathcal{H}}\left((\gamma_{\mathcal{H}}^{-1})^{ij}\psi_N(\gamma_{\mathcal{H}})\right), D_{\mathcal{H}}F_j\rangle\,|\mathcal{G}\right]$$

$$= E\left\{\phi(F)\sum_{j=1}^{m}\left[(\gamma_{\mathcal{H}}^{-1})^{ij}\psi_N(\gamma_{\mathcal{H}})\delta D_{\mathcal{H}}F_j - \langle D_{\mathcal{H}}\left((\gamma_{\mathcal{H}}^{-1})^{ij}\psi_N(\gamma_{\mathcal{H}})\right), D_{\mathcal{H}}F_j\rangle\right]|\mathcal{G}\right\}$$

$$= E(\phi(F)A_N|\mathcal{G}),$$

where A_N is some integrable random variable.

Assume that $F = (F_1, \dots, F_m)$ is rando m vector such that $F_i \in \mathbb{D}_{\infty}$for any $i = 1, \dots, m$.

Let $\mathcal{G} = \sigma\{G_i\,, i \geq 1\}$ be a countably smoothly generated σ- algebra such that the following condition holds:

(C) $\eta \in \mathbb{D}_{\infty}(H)$ implies $\prod_{\mathcal{H}}\eta \in \mathbb{D}_{\infty}(H)$.

This condition holds, for example, if the number of generators is finite, say $G_1, \dots, G_n$, $\left(det\langle DG_i, DG_j\rangle\right)^{-1} \in \cap_{p>1} L^p(\Omega)$ and $G_i \in \mathbb{D}_{\infty}$, $i = 1, \dots, n$.

Consider the partial Malliavin matrix of F, defined as before by

$$\gamma_{\mathcal{H}}^{ij} = \langle D_{\mathcal{H}}F_i, D_{\mathcal{H}}F_j\rangle.$$

Proposition (3): Let k be a positive integer. If $(\gamma_{\mathcal{H}}^{-1})^{ij} \in L^p$ for some $p > 4k$ and if we take $q > 1$ satisfying $\frac{1}{q} + \frac{4k}{p} < 1$, then the mapping

$$S(\mathbb{R}^m) \ni \phi \mapsto \phi(F) \in \mathbb{D}_{\infty}$$

is continuous with respect to the norm $\|\cdot\|_{-2k}$ on $S(\mathbb{R}^m)$, and the norm $\|\cdot\|_{p_0,-2k,\omega}$ on $\mathbb{D}_\infty$, for almost all ω, where $\frac{1}{p_0}+\frac{1}{q}=1$.

Proof: For $\phi\in S(\mathbb{R}^m)$ and $R\in\mathbb{D}_{q,2k}$, $\|R\|_{q,2k}\le 1$, we have, using Lemma (10),

$$\left|\int_\Omega \phi(F)(y)R(y)p(\omega,dy)\right| = \left|\int_\Omega (\{(1+|x|^2-\Delta)^k(1+|x|^2-\Delta)^{-k}\phi\})(F)(y)R(y)p(\omega,dy)\right|$$

$$=\left|\int_\Omega ((1+|x|^2-\Delta)^{-k}\phi)(F)(y)B_{2k}(R)(y)p(\omega,dy)\right|$$

$$\le \|(1+|x|^2-\Delta)^{-k}\phi\|_\infty E(|B_{2k}(R)|\ \ |\mathcal{G})$$

$$\le \|\phi\|_{-2K}E(|B_{2k}(R)|\ \ |\mathcal{G}),$$

for almost all ω.

Taking countable and dense subsets of $S(R^m)$ and $\mathbb{D}_{q,2k}$, we may assume that the above inequality holds for all ϕ and , a.s., and this concludes the proof.

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