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# SOME CLASSES OF LINEAR OPERATORS AND APPLICATIONS

**((Research Summary**

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# Contents

**Abstract**

*The study of generalized derivations are known as a focus of interest for many researchers and mathematicians in terms of operator theory. It has developed with the development of links between it and (mathematical) physics and mechanics, and it still attracts pure and applied mathematicians to study and analyze it. As well, generalized derivations are known as an essential component of operator theory, and there are many substantial and attractive results about their properties.*

*This thesis aims to study the spectral properties of certain classes of abnormal operators and apply this study to them through:*

*1. Verify the Fuglede-Putnam Theorem*

*2. Realization of band orthogonally and derivation generalized kernel induced by some abnormal classes of operators including posinoral factors and (p, k) - quasi-normal operators.*

***Keywords: (Derivations, Operator theory, Generalized derivations, Orthogonally, Derivation generalized kernel, Posinoral factors, (p, k)-quasi-normal operators)***

## 1. Research Introduction

Suppose B (H) is the set of all constrained linear operators operating on a complex Hilbert space H. The generalized derivation $\delta_{A, B}$ caused by the operator A, B ∈ B (H) on B (H) is defined by $\delta_{A, B}(X) = AX - XB$, X ∈ B (H).

If A = B, we indicate $\delta_{A, A}$ by $\delta_A$ and $\delta_A(X) = AX - XA$ is called the inner derivation. Generalized derivations are an important component of operator theory and there are many substantial conclusions about their properties.

The aim of this research is to attend some results about the orthogonality of the field and the kernel of generalized derivations. In addition, to study the properties of some unnatural classes of operators. Then, link them to the Fuglede-Putnam theory applied to these unnatural operators (posinormal and (p, k) - cosposonian natural factors).

In this section, some basic concepts and facts about Hilbert space and constrained operators are presented with some properties that we need in the sequel. In addition, we present the spectral characteristics of compact operators.

## 1.1 Hilbert Spaces

Hilbert Spaces are inner products on a linear space (vector space) H is a function h., .i from H × H into the field of scalars R (or C), which satisfies the following properties:

1.⟨ $\chi$ , $\chi$ ⟩ ≥ 0 for all $\chi$∈H.

2.⟨ $\chi$ , $\chi$ ⟩ = zero iff $\chi$ = zero.

3. ⟨ $\alpha$ x + $\beta$ y, z⟩ = $\alpha$ ⟨x, z⟩ + $\beta$ ⟨y, z⟩ for all x,y,∈H and any scalars α, β.

H is noticed to be a real or complex inner product space if h., .i is real or complex, respectively.

### 1.1.1 Remark

1. ⟨X, αy + βz⟩ = ⟨x, αy⟩ + ⟨x, βz⟩ = ¯α⟨x, y⟩ + β¯⟨x, z⟩ for all x, y, z in H and any scalars α, β ∈ C.

2. ⟨x, 0⟩ = 0 for all x ∈ H.

### 1.1.2 Definition

An inner product space on H defines a norm on H given by $||x|| = \sqrt{\langle x, x\rangle}$.

**1.1.3 Theorem** (Cauchy-Schwarz inequality)

For any x, y in an inner product space H, we have

$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$ or equivalently, $|\langle x, y\rangle| \leq ||x|| \, ||y||$.

**1.1.4 Remark**

Let H be an inner product space with a norm k.k, then

1. $||X|| \geq 0$ for all x in H and $||x|| = 0$ iff x = 0.

2. $||\alpha x|| = |\alpha| \, ||x||$ for all $x \in H$ and $\alpha \in C$.

3. $||X + Y|| \leq ||x|| + ||y||$ for all $x, y \in H$ (the triangle inequality).

**1.1.5 Definition**

A Hilbert space is defined as a total inner product space and every Hilbert space is a Banach space (i.e., every Cauchy sequence $(x_n)$ in H is approximate with respect to the norm induced by the inner product).

**1.1.6 Remark**

The function $||.||$ Is continuous. (i.e., if $(x_n)$ is a sequence in a Hilbert space H such that $x_n \to x$ in the norm topology, then $||x_n|| \to ||x||$ as $n \to \infty$).

**1.1.7 Definition**

Two vectors x and y in an inner product space H are called orthogonal (denoted $x \perp y$) if $\langle x, y \rangle$ = zero.

**1.1.8 Definition**

Let $\Omega$ be a subset of a Hilbert space H. Then the orthogonal complement of $\Omega$ (denoted by $\Omega\perp$) is defined by

$$\Omega \perp = \{y \in H: \langle x, y \rangle = \text{zero, for all } x \in \Omega\}.$$

**1.1.9 Remark**

1. $\Omega\perp$ is a closed subspace of H.

2. $\Omega \subseteq \Omega \perp\perp$.

3. $\Omega\perp\perp$ is the smallest closed subspace that contains $\Omega$.

**1.1.10 Theorem**

If $\Omega$ is a closed subspace of a Hilbert space H, then H can be written as the direct sum of $\Omega$ and $\Omega \perp$ denoted as $H = \Omega \oplus \Omega \perp$ (i.e., each $x \in H$ can be written uniquely as $x = x_1 + x_2$ where $x_1 \in \Omega$ and $x_2 \in \Omega \perp$).

## 1.2 Operators on Hilbert Spaces

In this section, the main basic characteristics of Hilbert space operators will be shown;

### 1.2.1 Theorem

Suppose H, K and L be Hilbert spaces, it will be as follows:

(a) If $A, B \in B(H, K)$, then $A + B \in B(H, K)$ and $||A + B|| \leq ||A|| + ||B||$.

(b) If $\alpha \in C$ and $A \in B(H, K)$, then $\alpha A \in B(H, K)$ and $||\alpha A|| = |\alpha| ||A||$.

(c) If $A \in B(H, K)$ and $B \in B(K, L)$, then $BA = (B \circ A) \in B(H, L)$ and $||BA|| \leq ||B|| \, ||A||$.

### 1.2.2 Definition

If $A \in B(H, K)$, then $B \in B(K, H)$ satisfying the equation $\langle Ax, y \rangle = \langle x, By \rangle$ for all $x \in H$, $y \in K$ is called the adjoint of A and is denoted by $B = A^*$. Thus $\langle Ax, y \rangle = \langle x, A^* y \rangle$.

**1.2.3 Theorem**

If A, B ∈ B (H) and α ∈ C, then

(a) $(A + B)^* = A^* + B^*$.

(b) $(\alpha A)^* = \alpha A^*$.

(c) $(AB)^* = B^*A^*$.

(d) $A^{**} = (A^*)^* = A$.

(e) $||A|| = ||A^*|| = ||A^*A||^{1/2} = ||AA^*||^{1/2}$.

**1.2.4 Definition**

An operator A ∈ B (H) is invertible if there exists an operator B in B (H) such that AB = BA = I, where I is the identity operator. B is unique and called the inverse of A, and is denoted by $A^{-1}$.

**1.2.5 Remark**

If A, B ∈ B (H) are invertible, then

1. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.

2. AB is invertible and $(AB)^{-1} = B−1A^{-1}$.

3. $An = A \circ A \ldots \circ A$ is invertible and $(An)^{-1} = A^{-n} = (A^{-1})^{n}$ for n = 1, 2,

4. $\alpha A$ is invertible and $(\alpha A)^{-1} = 1/\alpha A^{-1}$ for all $\alpha$ 6≠ 0.

5. $A^*$ is invertible and $(A^*)^{-1} = (A^{-1})^*$.

**1.2.6 Definition**

If $A \in B(H)$, we define

1. The kernel of A by Ker A = $\{x \in H: Ax = \text{zero}\}$.

2. The range of A by Ran A = $\{Ax: x \in H\}$.

* It can be noticed that Ker A and Ran A are subspaces of H.

**1.2.7. Definition**

If $A \in B(H)$, then:

(a) A is Hermitian (or self-adjoint) if $A^* = A$.

(b) A is unitary if $A^*A = AA^* = I$.

(c) A is normal if $A^*A = AA^*$.

(d) A is positive if $\langle A\chi , \chi \rangle \geq 0$ for all$\chi \in$H.

(e) A is an isometry if $||A\chi|| = ||\chi||$ for all $\chi \in H$.

**1.2.9 Remark**

1. Every positive operator is Hermitian.

2. Hermitian or unitary operators are normal; however, the converse is not true.

**1.3 Compact operators**

In this section, we discuss the basic characteristics of compact operators on a Hilbert space.

**1.3.1 Definition**

Compact operators are an operators $K \in B$ (H) is called compact if for each sequence ($\chi_n$) of unit vectors in H; the sequence ($K\chi_n$) has a convergent subsequence. If $K \in B$ (H) is compact and ($z_n$) is a bounded sequence in H, then ($Kz_n$) has a convergent subsequence.

**1.3.2 Theorem**

Suppose K, L be two compact operators in B (H), then:

(i) K + L is compact.

(ii) If $A \in B(H)$, then KA and AK are compact. (i.e., the set of compact operators is a two-sided ideal in B (H)).

**1.3.3 Theorem**

An operator $A \in B(H)$ is compact iff $A^*$ is compact.

**1.3.4 Theorem**

If $\{K_n\}$ is a sequence of compact operators in B (H and $||K_n - K|| \to 0$ as $n \to \infty$ where $K \in B(H)$, then K is compact.

**1.3.5 Theorem**

An operator $A \in B(H)$ is compact iff $A^*A$ is compact iff $|A|$ is compact.

**1.3.6 Theorem**

(The Hilbert-Schmidt Theorem) let A be a Hermitian compact operator in B (H). Then there exists an orthonormal basis $\{\varphi_n\}$ for H, so that $A\varphi_n = \lambda_n\varphi_n$ and $\lambda_n \to 0$ as $n \to \infty$. (i.e., A has an orthonormal basis of eigenvectors).

**2. Normal operators**

An operator $T \in B(H)$ is called normal if $TT^* = T^*T$. This is equivalent to $||T^*\chi|| = ||T\chi k||$ for all $\chi \in H$.

In this section, we shall introduce the basic facts about normal operators, which are going to be used in the sequel.

**2.1 Theorem**

If $T \in B(H)$ is normal, then $r(T) = ||T||$

Proof. Suppose $S = T^*T$. Since T is normal, then S is self-adjoint and from the inequality.

$$||S\chi||^2 = \langle S\chi, S\chi\rangle = \langle S^2\chi, \chi\rangle \leq ||S^2||\ ||\chi||^2$$

We get $||S||^2 \leq ||S^2||$. The opposite inequality follows from the fact $||S^2|| \leq ||S||.||S|| = ||S||^2$

Thus, we have the equality $||S^2|| = ||S||^2$ and by induction we obtain $||S^{2m}|| = ||S||^{2m}$, $\forall m$.

**2.2 Definition**

Let $T \in B(H)$, the set $W(T) := \{\langle T\chi, \chi\rangle : ||\chi|| = 1\}$

Is called the numerical range of T and the numerical radius of T is given by $\omega(T) := \sup\{|\lambda| : \lambda \in W(T)\}$.

An essential fact about W (T) is that it is always convex [17]. Moreover, the smallest closed convex set containing σ (T), denoted by conv σ (T), is contained in W (T).

**2.3 Proposition**

Assume T∋ B (H) is a regular operator. If T $\chi = \lambda\chi$, then T $^*\chi = \bar{\lambda}\chi$

**2.2 Classes of operators containing normal operators**

Other classes of operators, which contain many of the spectral properties of normal operators, contain the normal operator class. The hypo normal operator's class is also included.

**2.2.1 Definition**

An operator T ∈ B (H) is called hypo normal if $T^*T \geq TT^*$. Equivalently, $||T^*\chi|| \leq ||T\chi||$ for all χ ∈ H. In addition, if $A^*$ is hypo normal, A∋B (H) is referred to as co-hypo normal. This condition indicates that a normal operator is hypo normal.

**2.2.2 Proposition**

Assume T ∈ B (H) is a hypo normal, then:

1. T − λI is hypo normal,

2. If T χ = λχ, then $T^*$χ = λχ⁻,

3. If T χ= λχ and T y = μy where λ ≠μ, then⟨ χ , y⟩= zero.

**2.2.3 Proposition**

Suppose T ∈ B (H) is hypo normal. Then zero ∈ σ ($T^*T - TT^*$).

**2.2.4 Proposition**

If T ∈ B (H) is hypo normal and M ⊂ H is invariant under T, then $T_M$ is hypo normal. If $T_M$ is normal, then M reduces T.

**2.2.5 Proposition**

Assume T ∈ B (H) is a hypo normal, and M = {χ ∈ H: T χ= λχ}. Then $T_M$ returns to normal, and M decreases T.

## 3. Posinormal operators

In this section, we will go over posinormal and (p, k)-quasiposinormal operators, which are two new types of operators. Posinormal operators are a subclass of (p, k)-quasiposinormal operators and a superclass of normal and hypo normal operators (BEIBA, 2021).

If A ∈ B(H), then A is posinormal if there is a positive operator P B (H) such that $AA^* = A^*P A$. P(H) denotes the set of all posinormal operators on H, and P is known as the interrupter (Bonyo, 2011).

The related interrupter P must satisfy the criterion $||P|| \geq 1$ if the posinormal operator A is nonzero. Since,

$$||A||^2 = ||AA^*|| = ||A^*PA|| \leq ||A^*|| \, ||P|| \, ||A|| = ||P|| \, ||A||^2$$

**3.1 Posinormality versus hypo normality**

Posinormality does not imply hypo normality, but the Cesaro matrix and unilateral shift experience suggests the possibility of the contrary consequence (Bonyo, 2011).

For A, B ∈ B (H) the following propositions are identical:

1. Ran A ⊆ Ran B.

2. $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$; and

3. There exists T ∈ B (H) such that A = BT.

**3.2 Invertibility and translates**

### 3.2.1 Theorem

Every posinormal operator is invertible.

### 3.2.2 Corollary

Every invertible operator has the property of being coposinormal.

### 3.2.3 Corollary

Assume $A \in B(H)$, and $\lambda \notin \sigma(A)$ the spectrum of A. Then $A-\lambda I$ is posinormal.

It is self-evident that if T is hypo normal, then $\lambda T$ and $T + \lambda I$ are also hypo normal. For posinormal operators, the following theorem considers the same issues.

### 3.3 (p, k)-Quasiposinormal operators

For any operator $A \in B(H)$, set, as usual $|A| = (A^*A)^{1/2}$ and $[A^*, A] = A^*A-AA^* = |A|^2 - |A^*|^2$ (the self-commutator of A) (A. Bachir, 2016)

1. A is said p-hypo normal if $|A|^{2p} \geq |A^*|^{2p}$.
2. A is quasihyponormal if $A^*(A^*A - AA^*)A \geq 0$.
3. A is k-quasihyponormal if $A^{*k}(A^*A - AA^*)A^k \geq 0$
4. A is (p, k)-quasihyponormal if $A^{*k}((A^*A)^p - (AA^*)^p)A^k \geq 0$
5. A is posinormal if $|A^*|^2 \leq c^2 |A|^2$ for some $c > 0$
6. A is p-posinormal if $(AA^*)^p \leq c^2 (A^*A)^p$ for some $c > 0$

7. A is k-posinormal if $A^{*k} (c^2 |A|^2 - |A^*|^2) A^k \geq 0$ for some $c > 0$.
8. A is said to be (p, k)-quasiposinormal if $A^{*k} (c^2 (A^*A)^p - (AA^*)^p) A^k \geq 0$, for some $c > 0$

## 4. Fuglede-Putnam's Theorem

A pair (A, B) of operators can be confirmed from the Fuglede-Putnam theorem, in case $A^* X = XB^*$ whenever $AX = XB$ for some X in B (H).

In this chapter, the Fuglede-Putnam theorem with generalizations in Hilbert-Schmidt case will be discussed.

### 4.1Properties of the $C_p$ classes

Let H be a separable Hilbert space, and $\{\varphi_n\}^{\infty}_{n=1}$ an orthonormal basis for H. Then for any positive operator $A \in B(H)$, define

$$\operatorname{tr} A := \sum_{n=1}^{\infty} \langle A\phi_n, \varphi_n \rangle$$

The function tr has the following properties:

1. The quantity tr is independent of the choice {φi} of orthonormal basis.

2. tr (A + B) = tr A + tr B.

3. tr (λA) = λ tr A for all λ ≥ 0.

4. tr $(UAU^{-1})$ = tr A, for any unitary operator U.

5. If 0 ≤ A ≤ B, then tr A ≤ tr B.

**4.2Hilbert-Schmidt operators and case**

A compact operator A is said to be a Hilbert-Schmidt operator. The operator equation AX = XB implies $A^*X = XB^*$ when A and B are normal (Fuglede- Putnam's Theorem). If X is of Hilbert-Schmidt class $C_2$, the assumptions on A and B can be relaxed: it suffices that A and $B^*$ be hypo normal, or that B be invertible with $||A||||B^{-1}|| < 1$.

**4.3 An extension of Fuglede-Putnam Theorem for Posinormal operators**

If A ∈ B (H) is hypo normal and $B^*$ ∈ B (H) is posinormal, then $\Gamma_{A, B}$ is posinormal.

**4.4 An extension of Fuglede-Putnam Theorem for (p, k)-quasiposinormal operators**

If A is (p, k)-quasiposinormal and $B^*$ is (p, k)-quasihyponormal, then $\Gamma_{A, B}$ is (p, k)-quasiposinormal.

## 5. Orthogonally of the range and kernel of the generalized derivation

Suppose E be a complex Banach space. We say that $y \in E$ is orthogonal to $x \in E$ if for all complex $\lambda$ there holds (Bachir, 2012).

$$|\chi + \lambda y| \geq ||\chi||$$

Note that if y is orthogonal to $\chi$, then $\chi$ need not be orthogonal to y, so the orthogonally in this sense is asymmetric.

$$||AX - XA + S|| \geq ||S||$$

The range of the inner derivation $X \rightarrow \delta_{A, A}(X) = AX - XA$ is orthogonal to its kernel.

### 5.1 Generalized Derivations Induced by Hypo normal operators

#### 5.1.1 Theorem

If $A \in B(H)$ is a hypo normal operator and N is a normal operator in $\{A\}'$ where $\{A\}' = \{X: AX = XA\}$, then for all $\lambda \in \sigma_p(N)$, $|\lambda| \leq ||AX - XA + N||$ for all $X \in B(H)$ (Rashid, 2016).

**5.1.2 Theorem**

If $A \in B(H)$ is hypo normal, then for any normal operator $N \in \{A\}'$ and any $T \in B(H)$, we have $||N|| \leq ||N + AT - TA||$

**5.1.3 Theorem**

If $A, B \in B(H)$ such that A is hypo normal and $B^*$ is hypo normal, then for all $T \in Ker(\delta_{A,B})$ and all $X \in B(H)$, $||T|| \leq ||T + AX - XB||$, that is, $Ran(\delta_{A,B})$ is orthogonal to $Ker(\delta_{A,B})$.

**5.3 Orthogonally and Fuglede-Putnam Property**

**5.3.1 Definition**

The pair of operators (A, B) in B (H) satisfies the $(F\,P)_{B(H)}$ property if $AX = XB$ implies $A^*X = XB^*$ for some $X \in B(H)$ (Bachir, 2013).

**5.3.2 Theorem**

Suppose (A, B) be a pair of operators verifying the $(F\,P)_{B(H)}$ property, then Ran $\delta_{A,B}$ is orthogonal to Ker $\delta_{A,B}$ (A. Bachir, 2016).

**5.3.3 Corollary**

Ran $\delta_{A,B}$ is orthogonal to Ker $\delta_{A,B}$ if A and $B^*$ are hypo normal.

**5.3.4 Theorem**

Suppose A, B be operators in B (H) and $S \in C_2$ (H). Then $||\delta_{A,B}(X) + S||^2_2 = ||\delta_{A,B}(X)||^2_2 + |S||^2_2$

**5.3.5 Corollary**

Suppose A, B be operators in B (H) and $S \in C_2$ (H).

And

$||\delta^*_{A,B}(X) + S||^2_2 = ||\delta^*_{A,B}(X)||^2_2 + ||S||^2_2$

If and only if either of the following holds (Hoxha, 2013).

1. A is hypo normal and $B^*$ is an invertible operator.
2. A is p-hypo normal and $B^*$ is an invertible p-hypo normal.
3. A is k-quasihyponormal and $B^*$ is an invertible k-quasihyponormal.
4. A is p-quasihyponormal and $B^*$ is an invertible p-quasihyponormal.
5. A is (p, k)-quasiposinormal and $B^*$ is an invertible operator.

## 6. Research Conclusion

This part of research will include a summary of the main points of linear operators and applications. Besides, this chapter also will offer further suggestion for future related studies.

A group of new factors, the hypo-normal and (p, k)-quasiposinormal class, will be recognized as an extension of the sub-normal, p-post normal, and b-quasiposinormal class. In addition, the Fuglede-Putnam theorem for (p, k)-quasiposinormal has been demonstrated in the case of Hilbert-Schmidt space C2. As a result, it is observed that the range of generalized derivation caused by these classes of operators is orthogonal to the kernel.

In future research, it is supposed to find conditions, in order to get the results of Vogel led-Putnam theorem and orthogonally of Hilbert space.

## Bibliography

A. Bachir, A. (2016). An asymmetric Putnam-Fuglede Theorem for (p, k)-quasiposinormal operators. *Int. J. Contemp. Math. Sciences, 11*(4), 165-172.

Bachir, A. (2012). Range Kernel orthogonality of generalized derivations. *Int. J. Open Problems Compt. Math., 5*(4), 1-6.

Bachir, A. (2013). An extension of Fuglede-Putnam Theorem for certain posinormal. *Int. J. Contemp. Math. Sciences, 17*(8), 827-832.

BEIBA. (2021). POWER-POSINORMAL OPERATORS. *Methods of Functional Analysis and Topology, 27*(1), 18-24.

Bonyo, A. (2011). Norm of a Derivation and Hyponormal operators. *Int. Journal of Math. Analysis, 14*(4), 687 - 693.

Hoxha. (2013). Spectral Properties, Contractions and Fuglede-Putnam Theorem for A∗n operators. *Int. Journal of Math. Analysis, 7*(63), 3097 - 3108.

Rashid. (2016). PARANORMAL OPERATORS. *Commun. Korean Math. Soc, 31*(3), 549–565.